Math 135A, Fall 2019
December 09 or 13, 2019

FINAL EXAM (sample)

NAME(use CAPITAL letters, *first name first*):____

NAME(sign):________________________________

SECTION:_________________________

ID#:_______________________

Instructions: Each of the 6 problems has worth either 33 or 34 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. You can use the Formula Sheet provided at our course website. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 8 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	over 200

Before starting: Imagine that this is your exam and solve it in 2 hours. Problems 1, 2 and 3 come with solutions, so solve them on a different paper. **Avoid reading the solutions**, and compare them with yours only when you finish your simulation.

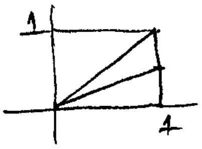
1. The pair (X, Y) of random variables has joint density given by $f(x,y) = \begin{cases} c(xy+1) & \text{if } x, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

(a) Determine c. (Recall $\int_0^1 x^n\,dx = 1/(n+1)$ for $n > -1$.)

$$c \int_0^1 dx \int_0^1 (xy+1)\,dy = c\int_0^1 dx\,\left(x\cdot\tfrac{1}{2}+1\right)$$

$$= c\left(\tfrac{1}{2}\cdot\tfrac{1}{2}+1\right) = c\cdot\tfrac{5}{4} \qquad c = \tfrac{4}{5}$$

(b) Determine $P(2Y \leq X | Y \leq X)$. $= \dfrac{P(Y \leq \frac{X}{2})}{P(Y \leq X)}$



$$= \frac{\frac{4}{5}\int_0^1 dx \int_0^{x/2} (xy+1)\,dy}{\frac{4}{5}\int_0^1 dx \int_0^{x} (xy+1)\,dy} = \frac{\int_0^1 dx\left(x\cdot\frac{x^2}{8}+\frac{x}{2}\right)}{\int_0^1 dx\left(x\cdot\frac{x^2}{2}+x\right)}$$

$$= \frac{\frac{1}{8}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}} = \frac{\frac{1}{16}+\frac{1}{2}}{\frac{1}{4}+1} = \frac{9}{20}$$

(c) Determine the density of the random variable $Z = \sqrt{X}$.

$$P(Z \leq z) = P(X \leq z^2) = \frac{4}{5}\int_0^{z^2} dx \int_0^1 (xy+1)\,dy$$

$z \in [0,1]$

$$= \frac{4}{5}\int_0^{z^2}\left(\tfrac{1}{2}x+1\right)dx$$

$$f_Z(z) = \frac{4}{5}\left(\tfrac{1}{2}z^2+1\right)\cdot 2z = \frac{4}{5}(z^3+2z) \quad \text{if } z \in [0,1]$$

$= 0$ otherwise

2. Shuffle a full deck of 52 cards.

(a) What is the probability that all Aces are together in the deck (i.e., the four Aces are four consecutive cards, in any order)?

$$\frac{4!\ 49!}{52!} \qquad \left(\text{or} \quad \frac{49}{\binom{52}{4}}\right)$$

(b) What is the probability that all the Aces are together in the deck and so are Kings, Queens, and Jacks.

$$\frac{(4!)^4 \cdot 40!}{52!}$$

(c) What is the probability that all the Aces are together in the deck and so are all the hearts ($\heartsuit$) cards?

$\boxed{\quad\heartsuit\quad}\boxed{A\heartsuit}\boxed{A}$ or $\boxed{A}\boxed{A\heartsuit}\boxed{\quad\heartsuit\quad}$

$52 - 16 = 36$ other cards

$$\frac{2 \cdot 3!\ 12! \cdot 37!}{52!}$$

Problem 2, continued.

(d) What is the probability that the first ~~four~~ three cards are all hearts?

$$\frac{1}{4}\cdot\frac{12}{51}\cdot\frac{11}{50} \qquad \left(\text{or} \quad \frac{\binom{13}{3}}{\binom{52}{3}}\right)$$

(e) What is the probability that the third card is of different suit than either of the first two cards? (For example, this event happens if the suits are, in order, ♡♡♢ or ♡♠♢, but not in the case ♡♠♡.)

P(1st two card same suit, 3rd different suit)

+ P(1st three cards diff. suits)

$$= \underset{\substack{\uparrow \\ \text{choose suit} \\ \text{of 1st 2 cards}}}{4}\cdot\frac{1}{4}\cdot\frac{12}{51}\cdot\frac{39}{50} + \underbrace{4\cdot3\cdot2}_{\substack{\uparrow \\ \text{suits of} \\ \text{1st 3 cards}}}\;\frac{1}{4}\cdot\frac{13}{51}\cdot\frac{13}{50}$$

3. Roll five red dice and five blue dice. All dice are fair.

(a) Let X be the number of 6's rolled on red dice and Y the total number of 6's. Identify the probability mass functions of X and Y. Compute $\mathrm{Var}(Y)$.

X is Binomial$(5, \frac{1}{6})$

Y is Binomial$(10, \frac{1}{6})$

$$\mathrm{Var}(Y) = 10 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{50}{36}$$

(b) Let X and Y be as in (a). Determine the joint probability mass function of X and Y. (Give a formula rather than a table.)

$$P(X=x, Y=y) = \underbrace{\binom{5}{x}\binom{5}{y-x}}_{\text{positions of 6's}} \underbrace{5}_{\text{other nos.}}{}^{10-y} \cdot \frac{1}{6^{10}}$$

$x = 0, \ldots, 5$

$x \leq y \leq x+5$

(c) Give the conditional probability $P(X=1|Y=1)$ as a single fraction. Compute also the unconditional probability $P(X=1)$. Are X and Y independent?

$P(X=1 \mid Y=1) = \frac{1}{2}$ (the 6 needs to appear on a red die)

$$P(X=1) = 5 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^4 = \frac{5^5}{6^5}$$

No. The two are different.

4. Suppose that you toss 2 fair coins. Let X be the number of heads on the first coin and Y the total number of heads.

(a) Compute the joint density of X, Y. Are X and Y independent?

$X \backslash Y$	0	1	2	f_X
0	1/4	1/4	0	1/2
1	0	1/4	1/4	1/2
f_Y	1/4	1/2	1/4	1

$X \sim Ber(1/2)$, $Y \sim Bin(2, 1/2)$

X, Y are dependent, since

$f(1,0) \neq f_X(1) f_Y(0)$

(b) Find $\rho(X, Y)$ and $\mathbb{E}(X|Y)$.

$\rho(X,Y) = Cov(X,Y) / \sqrt{Var(X)\, Var(Y)} = \cdots$ Do it!

$\mathbb{E}(X|Y=y) = 0\, f_{X|Y}(0|y) + 1\, f_{X|Y}(1|y) = f_{X|Y}(1|y).$

So, $\mathbb{E}(X|Y=0) = f_{X|Y}(1|0) = \dfrac{f(1,0)}{f_Y(0)} = \dfrac{0}{1/4} = 0.$

Similarly $\mathbb{E}(X|Y=1) = 1/2$ and $\mathbb{E}(X|Y=2) = 1$.

(c) Calculate the characteristic function $\phi_W(t)$, where $W = Y + 1$.

$Y \in \{0,1,2\} \Rightarrow W \in \{1,2,3\}$. $\mathbb{P}(W=k) = \mathbb{P}(Y=k-1)$
$k = 1, 2, 3.$

$$\phi_W(t) = \mathbb{E}(e^{itW}) = e^{it1}\, \mathbb{P}(W=1) + e^{it2}\, \mathbb{P}(W=2) + e^{it3}\, \mathbb{P}(W=3)$$

$$= \frac{1}{4} e^{it} + \frac{1}{2} e^{2it} + \frac{1}{4} e^{3it}.$$

5. Assume that X_1 and X_2 are independent $Exp(1)$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$.

(a) Compute the joint density of Y_1, Y_2.

(b) Find F_{Y_2} and $\mathbb{P}(Y_2 \geq 1)$.

(c) Find f_{Y_1} and $\mathbb{E}(Y_1|Y_2)$.

6. Determine whether the following statements are True or False (proof or counterexample).

(a) If $X \sim \mathrm{Ber}(p)$ is independent of $Y \sim \mathrm{Bin}(n,p)$, then $X+Y \sim \mathrm{Bin}(n+1,p)$.

See solutions to Midterm 2 morning

(b) If $\mathbb{E}(X) = 2$ and $\mathbb{E}(X^2) = 4$, then $\mathbb{E}[(X-2)^2] = 2$.

$\mathbb{E}((X-2)^2) = \mathbb{E}((X-\mathbb{E}X)^2) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = 4 - 2^2 = 0$

or $\mathbb{E}(X^2 - 4X + 4) = 4 - 4 \cdot 2 + 4 = 0.$ False.

(c) If X and Y are independent $\mathrm{Ber}(1/2)$, then $X|(X+Y=1) \sim \mathrm{Ber}(1/2)$.

See solutions to Midterm 2 afternoon

(d) If $X \sim \beta(2,2)$, then $\mathbb{E}(X^n) = \dfrac{1}{(n+1)!}$ for all $n \in \mathbb{N}$. $B(2,2) = \dfrac{\Gamma(2)\Gamma(2)}{\Gamma(2+2)} = \dfrac{1}{6}$

So $f_X(x) = \frac{1}{B(2,2)} x^{a-1}(1-x)^{b-1} = 6x(1-x)$, $x \in (0,1)$.

$\mathbb{E}(X^n) = \int_0^1 x^n \cdot 6x(1-x)\,dx = 6\left(\int_0^1 x^{n+1}dx - \int_0^1 x^{n+2}dx\right)$

$= 6\left(\frac{1}{n+2} - \frac{1}{n+3}\right) \neq \frac{1}{(n+1)!}$ False. for $n \geq 2$.

(e) If $f_X(x) = \frac{1}{2}e^{-|x|}$, for $x \in \mathbb{R}$, then the characteristic function of X is $\phi_X(t) = \dfrac{1}{t^2+i}$.

$\frac{1}{t^2+i}$ is NOT a characteristic function, since at $t=0$ we get $\frac{1}{0^2+i} = -i \neq 1$. False.