Fixation for two-dimensional $\mathcal{U}$-voter dynamics

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Abstract

Given a finite family $\mathcal{U}$ of finite subsets of $\mathbb{Z}^d \setminus \{0\}$, the *$\mathcal{U}$-voter dynamics* in the space of configurations $\{+,-\}^{\mathbb{Z}^d}$ is defined as follows: Every $v \in \mathbb{Z}^d$ has an independent exponential random clock, and, when the clock at v rings the vertex v chooses $X \in \mathcal{U}$ uniformly at random. If the set $v + X$ is entirely in state $*$ (where $* \in \{+,-\}$), then the state of v becomes $*$, otherwise nothing happens. The *critical probability* $p_c^{\text{vot}}(\mathbb{Z}^d,\mathcal{U})$ for this model is the infimum over p such that this system almost surely fixates at $+$ when the initial states for the vertices are chosen independently to be $+$ with probability p and to be $-$ with probability $1-p$. We prove that $p_c^{\text{vot}}(\mathbb{Z}^2,\mathcal{U}) < 1$ for a wide class of families $\mathcal{U}$.

Keywords: Voter model, Ising model, Glauber dynamics, Bootstrap percolation.

1 Introduction

Given some spin dynamics on $\mathbb{Z}^d$, the *critical probability* for fixation is the infimum over $p \in [0,1]$ such that fixation at $+$ occurs when the initial states for the vertices are chosen independently to be $+$ with probability p and to be $-$ with probability $1-p$. For the zero-temperature Glauber dynamics of the Ising model, Fontes, Schonmann and Sidoravicius [6] showed that $p_c^{\text{Is}}(\mathbb{Z}^d) < 1$. Since, by symmetry between $+$ and $-$, $p_c^{\text{Is}}(\mathbb{Z}^d) \geq 1/2$, we can restate this by saying that there exists a *phase transition*. Recently, Morris [10] generalized these dynamics by considering any finite family $\mathcal{U}$ of finite subsets of $\mathbb{Z}^d \setminus \{0\}$ and defining the $\mathcal{U}$-Ising dynamics (see Section 1.1); he conjectured that for the so-called critical families (see Definition 1.2), this model also has a phase transition. In this article, we focus on the $\mathcal{U}$-voter dynamics (see Section 1.2), and show that in this case, for a wide class of such families, we have a phase transition.

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1.1 Motivation: Glauber dynamics of the Ising model

Let $\mathcal{U} = \{X_1, \ldots, X_m\}$ be an arbitrary finite family of finite subsets of $\mathbb{Z}^d \setminus \{0\}$. Given a configuration in $\{+,-\}^{\mathbb{Z}^d}$, we say that $X \in \mathcal{U}$ *disagrees with vertex* $v \in \mathbb{Z}^d$ if each vertex in $v + X$ has the opposite state to that of v. The $\mathcal{U}$*-Ising dynamics* on $\mathbb{Z}^d$ with states $+$ and $-$ were introduced by Morris [10] as follows:

- Every $v \in \mathbb{Z}^d$ has an independent exponential random clock with rate 1.
- When the clock at vertex v rings, if there exists $X \in \mathcal{U}$ which disagrees with v then v flips its state. Otherwise nothing happens.

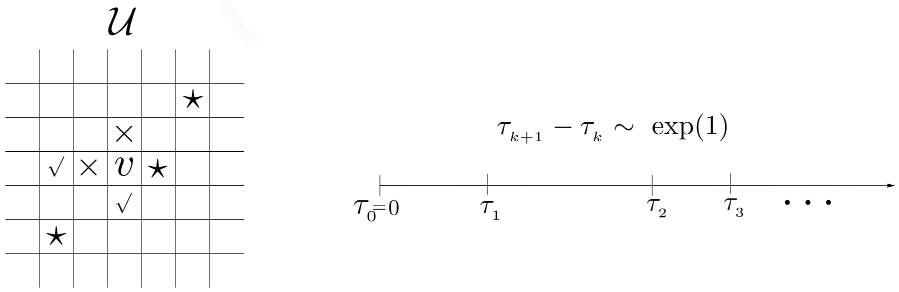

Figure 1: The clock at vertex v rings at time τ_k, then the state of v is updated by using the two-dimensional family $\mathcal{U} = \{X_1, X_2, X_3\}$, where $X_1 = \{-e_1, e_2\}$ is marked with $\times$, $X_2 = \{-2e_1, -e_2\}$ marked with $\surd$, and $X_3 = \{e_1, (2,2), (-2,-2)\}$ marked with $\star$.

Special cases of these dynamics have been extensively studied, for example, let $e_1, \ldots, e_d$ denote the canonical unit vectors in $\mathbb{Z}^d$, and consider the family $\mathcal{N}_r^d$ defined as the collection of all subsets of size $\geq r$ of $\{e_1, \ldots, e_d, -e_1, \ldots, -e_d\}$; when $\mathcal{U} = \mathcal{N}_d^d$ this process coincides with the so called *zero-temperature Glauber dynamics of the Ising model* (sometimes called *Metropolis dynamics*), see, for example [8].

Let $\sigma_t \in \{+,-\}^{\mathbb{Z}^d}$ denote the state of the system at time $t \geq 0$. Say that dynamics *fixate at* $+$ if for each vertex $v \in \mathbb{Z}^d$, there is a time $T_v \in [0, \infty)$ such that $\sigma_t(v) = +$ for all $t \geq T_v$, in other words, if the state of every vertex is eventually $+$. Now fix $p \in [0,1]$. We say that a set $A \subset \mathbb{Z}^d$ is *p-random* if it is chosen according to the Bernoulli product measure on $\mathbb{Z}^d$ (i.e. each of the sites of $\mathbb{Z}^d$ are included in A independently with probability p). Let the set $\{v \in \mathbb{Z}^d : \sigma_0(v) = +\}$ be chosen p-randomly and write $\mathbb{P}_p$ for the joint distribution of the initial spins and the dynamics realizations. We define the *critical probability* for the $\mathcal{U}$-Ising dynamics to be

$$p_c^{\mathrm{Is}}(\mathbb{Z}^d, \mathcal{U}) := \inf \{p : \mathbb{P}_p(\mathcal{U}\text{-Ising dynamics fixate at } +) = 1\}.$$

Write $p_c^{\mathrm{Is}}(\mathbb{Z}^d)$ for $p_c^{\mathrm{Is}}(\mathbb{Z}^d, \mathcal{N}_d^d)$. It was proved by Arratia [1] that $p_c^{\mathrm{Is}}(\mathbb{Z}) = 1$, basically, the reason is that for every $p \in (0,1)$, in dimension 1 every site changes state an infinite number of times. A well-known (and possibly folklore) conjecture states that $p_c^{\mathrm{Is}}(\mathbb{Z}^d) = 1/2$ for every $d \geq 2$. The first progress towards this conjecture was the following upper bound, proved by Fontes, Schonmann and Sidoravicius [6].

Theorem 1.1 (Fontes, Schonmann and Sidoravicius). *$p_c^{\mathrm{Is}}(\mathbb{Z}^d) < 1$ for every $d \geq 2$.*

Moreover, the authors in [6] showed that this fixation occurs in time with a stretched exponential tail. Morris [9] combined this theorem with some techniques from high dimensional bootstrap percolation (see Section 2.2) to prove that $p_c^{\mathrm{Is}}(\mathbb{Z}^d) \to 1/2$ as $d \to \infty$.

Another related result for the symmetric case $p = 1/2$ (physically in the Ising model setting, this corresponds to an initial quench from infinite temperature) is due to Nanda, Newman and Stein [11]. They proved that in dimension 2, every vertex almost surely changes state an infinite number of times, however, it is still unknown if the same holds for higher dimensions.

These dynamics have also been considered in other lattices. For instance, Damron, Kogan, Newman and Sidoravicius [5] considered slabs of the form $\mathbb{S}_k := \mathbb{Z}^2 \times \{0, 1, \ldots, k-1\}$ ($k \geq 2$) with the family $\mathcal{N}_3^3$. They proved a classification theorem which, surprisingly holds for *all* $p \in (0,1)$ and, in particular implies that $\mathbb{S}_k$ does not fixate at $+$ (however, each single vertex in $\mathbb{S}_2$ fixates at either $+$ or $-$). Therefore, in this particular setting which interpolates dimensions 2 and 3 (so Theorem 1.1 does not apply), the critical probability is 1 and there is no phase transition for fixation at $+$.

Now let's consider general $\mathcal{U}$ in dimension $d = 2$. For each $u \in S^1$ (the unit circle) we write $\mathbb{H}_u := \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}$ for the discrete half-plane whose boundary is perpendicular to u. In their groundbreaking work on general models of monotone cellular automata Bollobás, Smith and Uzzell [3] made the following important definitions:

Definition 1.2. The set $\mathcal{S}$ of *stable directions* is

$$\mathcal{S} = \mathcal{S}(\mathcal{U}) := \{u \in S^1 : X \not\subset \mathbb{H}_u, \ \forall X \in \mathcal{U}\}.$$

We say that $\mathcal{U}$ is *critical* if there exists a semicircle in S^1 that has finite intersection with $\mathcal{S}$, and if every open semicircle in S^1 has non-empty intersection with $\mathcal{S}$.

Morris [10] conjectured that for the critical models there exists a phase transition

Conjecture 1.3. *For every critical two-dimensional family $\mathcal{U}$, it holds that*

$$p_c^{\mathrm{Is}}(\mathbb{Z}^2, \mathcal{U}) < 1.$$

Note that the family $\mathcal{N}_2^2$ is critical because $\mathcal{S}(\mathcal{N}_2^2) = \{e_1, e_2, -e_1, -e_2\}$, and the conjecture holds for this family by Theorem 1.1.

One of the main difficulties of Conjecture 1.3 is that for the $\mathcal{U}$-Ising dynamics we do not know how to prove that droplets can be eroded in polynomial time (see Definition 1.6 and Section 3). For this reason we will instead focus on the $\mathcal{U}$-voter model where, as we will see, there is an additional bias in favor of the leading state that we will be able to exploit.

1.2 The $\mathcal{U}$-voter dynamics

Definition 1.4. Let $\mathcal{U} = \{X_1, \dots, X_m\}$ be an arbitrary finite family of finite subsets of $\mathbb{Z}^d \setminus \{0\}$. The *$\mathcal{U}$-voter dynamics* on $\mathbb{Z}^d$ with states $+$ and $-$ are defined as follows:

(a) Every $v \in \mathbb{Z}^d$ has an independent exponential random clock with rate 1.

(b) When the clock at v rings at (continuous) time $t \geq 0$, the vertex v *chooses* $X \in \mathcal{U}$ uniformly at random. If the set $v + X$ is entirely in state $*$ (where $* \in \{+, -\}$), then the state of v becomes $*$. Otherwise nothing happens.

For example, when $\mathcal{U}_{\text{LV}} = \{\{x\} : x \in U\}$ for some finite $U \subset \mathbb{Z}^d \setminus \{0\}$, in (b) the vertex v chooses some $x \in U$ independently with probability $1/|U|$, and then vertex v immediately adopts the same state as x. This is usually called a *linear voter model* [7]; of particular interest is the case where U consists of all $2d$ unit vectors in $\mathbb{Z}^d$. For related results see [7] an references therein.

The generator Ω of this Markov process acts on local functions f as

$$\Omega f(\sigma) = \sum_{v \in \mathbb{Z}^d} \frac{r_v(\sigma)}{m} [f(\sigma^v) - f(\sigma)],$$

where $r_v(\sigma)$ denotes the number of rules disagreeing with vertex v when the current configuration is σ, and σ^v is the configuration obtained from σ by flipping the state of vertex v. Observe that we have symmetry with respect to the interchange of the roles of $-$s and $+$s for these dynamics, and the system is monotone, namely, $r_v(\sigma)$ is increasing in σ when $\sigma(v) = -$ and decreasing in σ when $\sigma(v) = +$.

We are interested in the long-term behavior of this system, starting from a randomly chosen initial state, and ask whether the dynamics fixate or not. We remark that the families $\mathcal{U}_{\text{LV}}$ described above are not critical and, in fact, their dynamics do not fixate at $+$ (unless $p = 1$). For instance, if U consists of all $2d$ unit vectors then $\int_0^t \mathbb{1}\{\sigma_s(v) = -\}\, ds/t$ converges to $1 - p$ a.s. for $d \geq 2$ as $t \to \infty$ (see [4]), but if fixation at $+$ occurred then it should converge to 0. However, the critical families we are considering have a behavior quite different from that of $\mathcal{U}_{\text{LV}}$.

Let $p_c^{\text{vot}}(\mathbb{Z}^d, \mathcal{U})$, the *critical probability* of the $\mathcal{U}$-voter dynamics on $\mathbb{Z}^d$ to be

$$p_c^{\text{vot}}(\mathbb{Z}^d, \mathcal{U}) := \inf \{p : \mathbb{P}_p(\mathcal{U}\text{-voter dynamics fixate at } +) = 1\}. \tag{1}$$

To state our result we need some 2-dimensional definitions, so set $d = 2$.

Definition 1.5. Let $\mathcal{T} \subset \mathcal{S}$. A *$\mathcal{T}$-droplet* is a non-empty set of the form

$$D = \bigcap_{u \in \mathcal{T}} (\mathbb{H}_u + a_u),$$

for some collection $\{a_u \in \mathbb{Z}^2 : u \in \mathcal{T}\}$. When D is finite and has diameter (the maximum distance between two points in D) in $(L-1, L]$ we call D a *$(\mathcal{T}, L)$-droplet*.

We will always consider subsets $\mathcal{T} \subset \mathcal{S}$ such that D is finite (for example, when $\mathcal{U}$ is critical we can choose at least one such $\mathcal{T}$). Suppose that every vertex in a $(\mathcal{T}, L)$-droplet D is in state $-$, and every vertex outside D is frozen in state $+$ (see Figure 2). When we run the $\mathcal{U}$-voter dynamics, one might expect D to become entirely filled with $+$ in polynomial time in L.

Definition 1.6. Let D be a $(\mathcal{T}, L)$-droplet. Assume we start the process with D entirely occupied by states $-$, and all other states are $+$. The *droplet erosion time* $T(D)$ is the first time when D is fully $+$.

The droplet erosion time is well defined, because $\mathcal{T} \subset \mathcal{S}$, so the states outside D will never flip (see Figure 2), and eventually every state in D will become $+$ forever. In other words, by recurrence of finite state irreducible Markov chains it follows that $T(D) < \infty$ a.s..

Definition 1.7. We say that $\mathcal{U}$ is *poly-eroding* if we can choose $\mathcal{T} \subset \mathcal{S}$ and a constant $\eta > 0$ such that any $(\mathcal{T}, L)$-droplet D satisfies

$$\mathbb{P}_p(T(D) > L^\eta) \le e^{-L}, \tag{2}$$

for all L large enough.

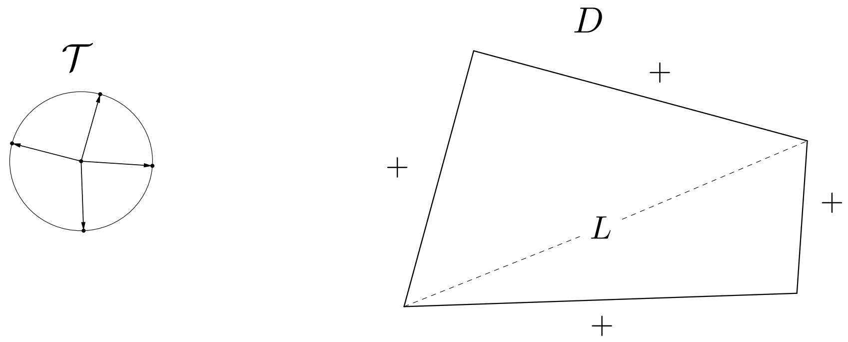

Figure 2: Four stable directions determining a droplet whose diameter depends on L

In fact, for every critical family it is easy to prove (2) with an exponential bound for $T(D)$ instead of polynomial in L. Moreover, numerical simulations suggest the following conjecture to be true, which is quite obvious but hard to prove:

Conjecture 1.8. *Every critical family is poly-eroding.*

Our main theorem is that in dimension $d = 2$, there exists a phase transition for some critical families (Conjecture 1.8 would imply it for *all* critical families).

Theorem 1.9. *If $\mathcal{U}$ is a poly-eroding critical two-dimensional family, then*

$$p_c^{\text{vot}}(\mathbb{Z}^2, \mathcal{U}) < 1. \tag{3}$$

In Section 3 we will prove that when a family has a *good* stable direction (see Definition 3.1), then it is poly-eroding. The following are examples of families having e_2 as a good direction:

Fix $a \in \mathbb{Z}^+$, $\mathcal{U}'$ a 2-dimensional family and $\mathcal{V}, \mathcal{W}$ 1-dimensional families such that

(I) $\forall X \in \mathcal{U}', X \cap \mathbb{H}_{-e_2} \neq \varnothing$ and $0 \in \mathrm{Conv}(X)$ (the convex hull of X),

(II) $0 < r^-(\mathcal{V}) \leq r^+(\mathcal{W})$ and $r^-(\mathcal{W}) \leq r^+(\mathcal{V})$, or $\mathcal{V} = \varnothing$ and $r^+(\mathcal{W})r^-(\mathcal{W}) > 0$,

where $r^+(\mathcal{V})$ (resp. $r^-(\mathcal{V})$) denotes the number of rules of $\mathcal{V}$ entirely contained in $\mathbb{Z}^+$ (resp. $\mathbb{Z}^-$). For instance, $\mathcal{V} = \{\{-3,-2,-1\},\{2\}\}$, $\mathcal{W} = \{\{-4\},\{1\},\{5\},\{-1,3\}\}$, and $\mathcal{U}' = \varnothing$. Then the following *induced* 2-dimensional family is poly-eroding:

$$\mathcal{U}_a(\mathcal{U}', \mathcal{V}, \mathcal{W}) := \mathcal{U}' \cup \bigcup_{Y \in \mathcal{V}} R(a, Y) \cup \bigcup_{Y \in \mathcal{W}} R(-a, Y),$$

here $R(a, Y)$ denotes the rule $\{(a, 0)\} \cup \{(0, y) : y \in Y\}$.
Known examples are $\mathcal{U}_1(\{\{1,-1\}\},\{\{1\},\{-1\}\},\{\{1\},\{-1\}\})$ which is just the family $\{\{e_2,-e_2\},\{e_1,e_2\},\{e_1,-e_2\},\{-e_1,e_2\},\{-e_1,-e_2\}\} = \mathcal{N}_2^2 \setminus \{\{e_1,-e_1\}\}$, and, in the literature, the family $\mathcal{U}_1(\{\{1,-1\}\},\varnothing,\{\{1\},\{-1\}\}) = \{\{e_2,-e_2\},\{-e_1,e_2\},\{-e_1,-e_2\}\}$ is usually called Duarte model (see [3]).

Conditions (I) and (II) are made to guarantee that $\{e_1,-e_1,e_2,-e_2\} \subset \mathcal{S}(\mathcal{U}_a(\mathcal{U}',\mathcal{V},\mathcal{W}))$. However, checking that e_2 is a good direction (even defining a good direction) involves more notation and therefore will be omitted here, but we will be able to deduce it by using Lemma 3.9 in Section 3.3, just by drawing the rules. One of the main ingredients in that proof will be the fact that every rule in $\mathcal{U}_a[\mathcal{U}',\mathcal{V},\mathcal{W}]$ has at most 1 vertex living in the x-axis, namely, either $(a,0)$ or $(-a,0)$. So, basically for this reason, the same lemma will serve to conclude that given $a_1,\ldots,a_k \in \mathbb{Z}^+$, the family

$$\mathcal{U}_{a_1}[\mathcal{U}_1',\mathcal{V}_1,\mathcal{W}_1] \cup \mathcal{U}_{a_2}[\mathcal{U}_2',\mathcal{V}_2,\mathcal{W}_2] \cup \cdots \cup \mathcal{U}_{a_k}[\mathcal{U}_k',\mathcal{V}_k,\mathcal{W}_k]$$

is poly-eroding, whenever $\mathcal{U}_i',\mathcal{V}_i,\mathcal{W}_i$ satisfy (I) and (II) for each $i \leq k$.

The rest of the paper is organized as follows: In Section 2 we give an sketch of the proof, define the *bootstrap percolation process* and state the properties we are going to use from it. Section 3 is the technical estimate for the single droplet erosion time, the inequality in Definition 1.6 is the only step we do not know how to prove for the $\mathcal{U}$-Ising dynamics. In Section 4 we couple the $\mathcal{U}$-voter dynamics with some block-dynamics and show that the probability that there exists $-$s spins in each block decreases sufficiently fast as time increases. In section 5 we prove that the probability that a block is '*influenced*' by *non-neighbors blocks* before it becomes entirely $+$ is sufficiently small, and then put the pieces together. Section 6 is dedicated to some future work.

2 Outline of the proof and bootstrap percolation

Here we give a sketch of the proof of Theorem 1.9. In fact, by using the techniques of [6], we will be able to prove an stronger result, namely, that fixation occurs in time with a stretched exponential tail.

Theorem 2.1. *There exist $\beta > 0$, $p_0 < 1$ such that for every $p > p_0$ it holds that*

$$\mathbb{P}_p[\sigma_t(0) = -] \leq \exp(-t^{\beta}), \tag{4}$$

for all sufficiently large t.

Once we have proved this statement, by using the Markov property we can find another constant $\beta' > 0$ such that $\mathbb{P}_p[\exists s \geq t : \sigma_s(0) = -] \leq \exp(-t^{\beta'})$ for t large enough, and Theorem 1.9 follows by applying Borel-Cantelli Lemma.

2.1 Sketch of Theorem 2.1

As that proof in [6], we use a multi-scale analysis; this consists of observing the process in some large boxes B_k at some times T_k which increase rapidly with k. This is done by induction on k; $T_0 = 0$ and suppose we are viewing the evolution inside the interval $[T_{k-1}, T_k)$. In B_k we couple the process with a *block-dynamics* which favors the $-$ team, in the sense that, when there is some $-$ in B_k at time T_k in the original process then it is also true for the block-dynamics. Inside B_k we allow the $-$ team to 'infect' the $+$ team via their own *bootstrap process* (see Section 2.2) meaning that just $+$ vertices are allowed to flip. We prove that up to time T_k every 'droplet' $D \subset B_k$ full of $-$s has 'relatively big' size with small probability. In other words, such droplets satisfy size$(D) \ll$ size(B_k) with high probability. Then we prove that before such droplets D could be created, the $+$ team inside B_k will typically eliminate it. The last step is to show that the probability that the $-$ team could receive any help from outside of B_k is also small.
At time T_k, if there is some $-$ in B_k we declare B_k to be a $-$, otherwise declare B_k to be $+$. Then we consider the evolution in a new time interval $[T_k, T_{k+1})$. The next step, we consider a larger box B_{k+1} consisting of several copies of B_k that we have declared to be either $-$ or $+$, and we start over again. By induction on k, we will show that if $q := 1 - p$ is very close to 0, Theorem 2.1 holds for all times of the form $t = T_k$. Finally, by using another coupling trick, we will extend the statement for all $t \geq 0$.

2.2 Bootstrap percolation families

First, we review a large class of d-dimensional monotone cellular automata, which were recently introduced by Bollobás, Smith and Uzzel [3], and then focus on dimension two.

Let $\mathcal{U}$ be an arbitrary finite family of finite subsets of $\mathbb{Z}^d \setminus \{0\}$. We call $\mathcal{U}$ the *update family*, each $X \in \mathcal{U}$ an *update rule*, and the process itself *$\mathcal{U}$-bootstrap percolation*. Now given a set $A \subset \mathbb{Z}^d$ of initially *infected* sites, set $A_0 = A$, and define for each $t \geq 0$,

$$A_{t+1} = A_t \cup \{x \in \mathbb{Z}^d : x + X \subset A_t \text{ for some } X \in \mathcal{U}\}.$$

Thus, a site x becomes infected at time $t+1$ if the translate by x of one of the sets of the update family is already entirely infected at time t, and infected sites remain infected forever. The set of eventually infected sites is the *closure* of A, denoted by $[A] = \bigcup_{t\geq 0} A_t$.

Set $d = 2$. Recall that for each $u \in S^1$, we denote $\mathbb{H}_u := \{x \in \mathbb{Z}^2 : \langle x, u\rangle < 0\}$. We say that u is a *stable direction* if $[\mathbb{H}_u] = \mathbb{H}_u$ and we denote by $\mathcal{S} = \mathcal{S}(\mathcal{U}) \subset S^1$ the collection of stable directions. Observe that this definition of $\mathcal{S}$ coincides with the one given in Definition 1.2. The following classification of two-dimensional update families was proposed by Bollobás, Smith and Uzzell [3].

An update family $\mathcal{U}$ is:

- *supercritical* if there exists an open semicircle in S^1 that is disjoint from $\mathcal{S}$;
- *critical* if there exists a semicircle in S^1 that has finite intersection with $\mathcal{S}$, and if every open semicircle in S^1 has non-empty intersection with $\mathcal{S}$;
- *subcritical* otherwise.

The justification for this trichotomy should become inspired by the next result. Suppose we performance the bootstrap percolation process on $\mathbb{Z}_n^2$ instead of $\mathbb{Z}^2$, $A \subset \mathbb{Z}_n^2$ is p-random, and consider the critical probability

$$p_c(\mathbb{Z}_n^2, \mathcal{U}) := \inf\{p : \mathbb{P}_p([A]_{\mathcal{U}} = \mathbb{Z}_n^2) \geq 1/2\}.$$

Bollobás, Smith and Uzzell [3] proved that the critical probabilities of supercritical families are polynomial, while those of critical families are polylogarithmic. Later, Balister, Bollobás, Przykucki and Smith [2] proved that the critical probabilities of subcritical models are bounded away from zero. We summarize those results in the following

Theorem 2.2 (2-dimensional classification)**.** *Let $\mathcal{U}$ be a 2-dimensional update family*

1. *If $\mathcal{U}$ is supercritical then $p_c(\mathbb{Z}_n^2, \mathcal{U}) = n^{-\Theta(1)}$;*
2. *If $\mathcal{U}$ is critical then $p_c(\mathbb{Z}_n^2, \mathcal{U}) = (\log n)^{-\Theta(1)}$;*
3. *If $\mathcal{U}$ is subcritical then* $\liminf p_c(\mathbb{Z}_n^2, \mathcal{U}) > 0$.

Remark 2.3. In this paper we will only deal with critical families, some reasons are

- The families $\mathcal{U}_{LV}$ are supercritical and do not fixate (see, for example, [4]).
- The family $\mathcal{N}_{d+1}^d$ is subcritical and we do not hope it will fixate at $+$, because any translate of $\{1,2\}^d$ that is entirely $-$ at time $t = 0$ will remain $-$ forever. It could be the case that some vertices will fixate at $+$ and others at $-$.

Given a poly-eroding family we will refer to its associated $\mathcal{T}$-droplet (see Definition 1.6) simply as *droplet*. Next we introduce an algorithm whose importance is to provide 2 key lemmas concerning droplets: an "Aizenman-Lebowitz lemma", which says that an covered droplet contains covered droplets of all intermediate sizes, and an extremal lemma, which says that an covered droplet contains a linear proportion of initially infected sites. Let us write diam(D) for the diameter of a droplet D.

Definition 2.4 (Covering algorithm)**.** Suppose n is large and $A \subset \mathbb{Z}_n^2$. The first step is to choose a sufficiently large constant κ, fix a droplet $\hat{D}$ of diameter roughly κ, and place a copy of $\hat{D}$ (arbitrarily) on each element of A. Now, at each step, if two droplets in the current collection are within distance κ of one another, then remove them from the collection, and replace them by the smallest droplet containing both. This process stops in at most $|A|$ steps with some finite collection of droplets, say $\{D_1, \dots, D_z\}$.

If a droplet occurs at some point in the covering algorithm, then let us say that it is *covered* by A. If κ is chosen sufficiently large, then one can prove that the final collection of droplets covers $[A]_{\mathcal{U}}$ (see [3] for details). Now we are ready to state the 2 key lemmas which help to control the expanding (see [3] for the proof).

Lemma 2.5 (Aizenman-Lebowitz lemma)**.** *Let D be a covered droplet. Then for every $1 \le k \le diam(D)$, there is a covered droplet $D' \subset D$ such that $k \le diam(D') \le 3k$.*

Lemma 2.6 (Extremal lemma)**.** *There exists a constant $\varepsilon > 0$ such that for every covered droplet D, $|D \cap A| \ge \varepsilon \cdot diam(D)$.*

3 The 1-dimensional approach

In this section we prove that once we can find an stable direction good enough then we can prove that the family is poly-eroding by using a 1-dimensional argument.

Instead of restricting the $\mathcal{U}$-voter dynamics to D by only freezing every vertex outside D in state $+$, now we will consider a particular evolution of the dynamics in dimension 1 by freezing everything except a finite segment orthogonal to some stable direction, and show how things can be deduced from this 1-dimensional setting.

3.1 A good stable direction

Fix $y = (y_1, y_2) \in \mathcal{S}$ a *rational direction*, it means either y_2/y_1 is rational or $y_1 = 0$. For each $L \in \mathbb{N}$, we let $\varrho = \varrho(y, L)$ be any fixed segment consisting of L consecutive vertices in the discrete line $\{x \in \mathbb{Z}^2 : \langle x, y\rangle = 0\}$. Now suppose we freeze each vertex in $\mathbb{H}_y$ in state $-$ and each state outside $\mathbb{H}_y \cup \varrho$ in state $+$, and at time $t = 0$ each vertex in ϱ has state $-$, then we let the dynamics evolve only on ϱ (see Figure 5).

Given a configuration $\eta \in \{+,-\}^{\varrho}$ denote η^+ (resp. η^-) the set of vertices in η having $+$ (resp. $-$) state.

Definition 3.1. 1. Given $y \in \mathcal{S}$ and $L \in \mathbb{N}$, for every $t \ge 0$ we let η_t to be the configuration in $\{+,-\}^{\varrho}$ at time t in these 1-dimensional dynamics.

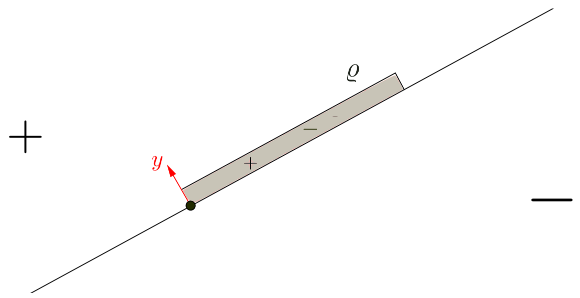

Figure 3: $\mathbb{H}_y$ entirely $-$ and $(\mathbb{H}_y \cup \varrho)^c$ entirely $+$

2. Say that $\mathcal{U}$ has a *good* direction $y \in \mathcal{S}$ if it is rational and, in the 1-dimensional dynamics, for each $L \in \mathbb{N}$ and $t \geq 0$ we have

$$\sum_{v \in \eta_t^+} r_v(\eta_t) \leq \sum_{u \in \eta_t^-} r_u(\eta_t). \tag{5}$$

Denote $\ominus$ (resp. $\oplus$) the configuration in $\{+,-\}^\varrho$ where all vertices are in state $-$ (resp. $+$), and observe that Condition (5) is trivial for $t = 0$ since $\eta_0 = \ominus$ and this gives LHS in (5) equals 0, also when t is large since then $\eta_t = \oplus$ (the segment fixates at $\oplus$) and for this configuration $y \in \mathcal{S}$ implies LHS=0 too.

Our aim now is to show that having a good direction is a sufficient condition to be poly-eroding. Given a good direction y, we are interested in

$$\tau = \tau(y, L) := \inf\{t : \eta_t = \oplus\}. \tag{6}$$

Proposition 3.2. *If $\mathcal{U}$ has a good direction y, then there is a constant $M > 1$ such that for L large enough we have $\mathbb{P}[\tau(y, L) > L^{M-1}] \leq e^{-1}$.*

We move the proof of this proposition to the next section. Recall that in Definition 1.6 we need to choose a nice subset of the stable set. This is the content of the next lemma:

Lemma 3.3. *Given $y \in \mathcal{S}$, there exists a finite set $\mathcal{S}_4 \subset \mathcal{S}$ such that $0 \in Conv(\mathcal{S}_4)$ (the convex hull of $\mathcal{S}_4$) and $y \in \mathcal{S}_4$.*

We will also prove this lemma in the next section. By combining these 2 sentences we can prove the

Proposition 3.4. *If $\mathcal{U}$ has a good direction then there exists a set $\mathcal{S}_4 \subset \mathcal{S}$ inducing a finite droplet D, and a constant $M > 1$ such that for every $t \geq 0$ it holds that*

$$\mathbb{P}_p[T(D) > tL^{M-1}] \leq Le^{-t}, \tag{7}$$

when L is large enough.

Proof. To fix ideas, we can assume that $\mathcal{U}$ has $y = (0, 1)$ as good direction, then we consider the set $\mathcal{S}_4$ given by Lemma 3.3. The induced $\mathcal{S}_4$-droplet D (see Definition 1.6) is finite because $0 \in Conv(\mathcal{S}_4)$, say D has diameter L, so that $T(D)$ is well defined.
We couple the dynamics with the following one: We allow to flip just the vertices at the first (top) line of the droplet, when they are all +s, we allow to flip just the vertices at the second line, and so on until we arrive at the bottom line. This coupled dynamic dominates the original one by monotonicity, and since the height of the droplet is at most L, then it is enough to show, by union bound, that for all t we have

$$\mathbb{P}_p[T_{coup} > tL^{M-1}] \leq e^{-t}, \tag{8}$$

where T_{coup} is the time to erode the top line in the coupled dynamic. Moreover we can assume that this top line has L vertices, since all lines have at most L vertices and having less vertices only helps to erode faster. To this end, let's consider the 1-dimensional process η_t given in Definition 3.1, because of the boundary conditions $T_{coup} \leq \tau(y, L)$ in distribution, so by Proposition 3.2

$$\mathbb{P}_p[T_{coup} > L^{M-1}] \leq \mathbb{P}[\tau(y, L) > L^{M-1}] \leq e^{-1}.$$

Finally, by Markov property it is straightforward to get (8). ♦

Corollary 3.5. *If a family $\mathcal{U}$ has a good direction, then it is poly-eroding.*

3.2 Filling the gaps

We start by proving Lemma 3.3, so we need some notation: write $[u, v]$ for the closed interval of directions between u and v (also (u, v) for the open interval). Say $[u, v]$ is *rational* if both u and v are rational directions. Our choice of $\mathcal{S}_4$ will depend on the following lemma:

Lemma 3.6. *The stable set $\mathcal{S}$ is a finite union of rational closed intervals of S^1.*

Proof. See [3]. ♦

With this tool, now we prove that in fact the set $\mathcal{S}_4$ in Lemma 3.3 can be chosen of size 3 or 4.

Proof. [**Lemma 3.3**] Let $y \in \mathcal{S}_4$ by definition. If $-y \in \mathcal{S}$ since $\mathcal{U}$ is critical we can choose $x \in (y, -y) \cap \mathcal{S}$, $z \in (-y, y) \cap \mathcal{S}$ and set $\mathcal{S}_4 = \{x, y, -y, z\}$. If $-y \notin \mathcal{S}$, then take $x \in \mathcal{S}$ in the open semicircle opposite to y, we can suppose wlog that $x \in (y, -y)$. Moreover, since $\mathcal{S}$ is closed by Lemma 3.6, we can choose x such that $\mathcal{S} \cap [x, -y] = \{x\}$. Then select $z \in \mathcal{S} \cap (x, -x)$ and observe that in fact $z \in \mathcal{S} \cap (-y, -x)$, so define $\mathcal{S}_4 = \{x, y, z\}$. In both cases, $0 \in \mathrm{Conv}(\mathcal{S}_4)$. ♦

Now we move to the proof of Proposition 3.2. To do so we use Markov's inequality $\mathbb{P}[\tau > \alpha] \leq \mathbb{E}[\tau]/\alpha$, so the first step is to show that if we can find a function on $\{+, -\}^{\varrho}$

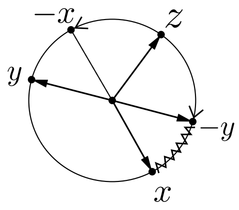

Figure 4: 3 or 4 stable directions

providing a bias in favor to the vertices in state + in the dynamics η_t (see Definition 3.1), then we can bound $\mathbb{E}[\tau]$ in terms of f and the extreme configurations $\ominus$ and $\oplus$.

Since $\varrho = \varrho(y, L)$ has L vertices can identify it with the initial segment $[L]$, so we can write the generator for the 1-dimensional process η_t as

$$\Omega f(\eta) = \sum_{v=1}^{L} \frac{r_v(\eta)}{m}[f(\eta^v) - f(\eta)].$$

Lemma 3.7. *Suppose there exists a function $f : \{+,-\}^L \to \mathbb{R}$ such that $\Omega f(\eta_t) \leq -1$ for all $t < \tau := \tau(y, L)$, then*

$$\mathbb{E}[\tau(y, L)] \leq f(\ominus) - f(\oplus). \tag{9}$$

Proof. Consider the martingale $M_t = f(\eta_t) - \int_0^t \Omega f(\eta_s)\, ds$, by optional stopping

$$\begin{aligned} f(\ominus) = \mathbb{E}[f(\eta_0)] &= \mathbb{E}[M_0] \\ &= \mathbb{E}[M_\tau] = \mathbb{E}[f(\eta_\tau) - \int_0^\tau \Omega f(\eta_s)\, ds] \\ &\geq \mathbb{E}[f(\eta_\tau)] + \mathbb{E}[\int_0^\tau 1\, ds] = f(\oplus) + \mathbb{E}[\tau], \end{aligned}$$

and the result follows. ♦

In order to apply this result, the next step is to show that if y is a good direction then we can define an explicit function f such that $f(\ominus) - f(\oplus)$ is polynomial in L.

Proposition 3.8. *If y is a good direction then there exists a function f satisfying the hypothesis in Lemma 3.7 such that RHS in (9) is $O(L^2)$.*

Proof. Set $h(0) = 0$ and for $k = 0, 1, \ldots, L-1$ define $h(k+1) = h(k) + m \cdot (L-k)$. So the function f is defined as follows: Given η with $k = k(\eta)$ vertices in state $-$ we set $f(\eta) = h(k)$. Observe that

$$f(\ominus) - f(\oplus) = h(L) - h(0) = \sum_{k=0}^{L-1}[h(k+1) - h(k)] = \sum_{k=0}^{L-1} m \cdot (L-k) = O(L^2).$$

Moreover, given $\eta = \eta_t$ with $k = k(\eta) \geq 1$ vertices in state $-$ we have

$$\begin{aligned}
m[\Omega f](\eta) &= \sum_{v=1}^{L} r_v(\eta)[f(\eta^v) - f(\eta)] \\
&= -\sum_{v\in\eta^-} r_v(\eta)[f(\eta) - f(\eta^v)] + \sum_{v\in\eta^+} r_v(\eta)[f(\eta^v) - f(\eta)] \\
&= -\sum_{v\in\eta^-} r_v(\eta)[h(k) - h(k-1)] + \sum_{v\in\eta^+} r_v(\eta)[h(k+1) - h(k)] \\
&= -\sum_{v\in\eta^-} r_v(\eta)[m(L-(k-1))] + \sum_{v\in\eta^+} r_v(\eta)[m(L-k)] \\
&\leq -\sum_{v\in\eta^-} r_v(\eta)m(L-(k-1)) + \sum_{v\in\eta^-} r_v(\eta)m(L-k) = -(\sum_{v\in\eta^-} r_v(\eta))m \\
&\leq -m.
\end{aligned}$$

So $\Omega f(\eta) \leq -1$ for all $\eta \neq \oplus$ and we are done. ♦

Finally, we are ready to conclude:

Proof. [**Proposition 3.2**] If $\mathcal{U}$ has a good direction y then apply Proposition 3.8 and then Lemma 3.7 to get $\mathbb{E}[\tau(y, L)] = O(L^2) \leq e^{-1}L^{M-1}$ for some constant $M > 3$, and for L large enough, so by applying Markov inequality we are finished. ♦

3.3 Examples and simulations

It is clear that the known families $\mathcal{N}_2^2$ and Duarte-like verify Condition (5). Now we give a criterion which will allow us to check if a family has a good direction just by drawing the rules in $\mathbb{Z}^2$. For $y \in \mathcal{S}$ consider the line $l_y := \{x \in \mathbb{Z}^2 : \langle x, y\rangle = 0\}$

Lemma 3.9 (Drawing)**.** *Suppose there exists a rational direction $y \in \mathcal{S}$ such that*

(a) Every $X \in \mathcal{U}$ has at most 1 vertex in l_y.

(b) Fix $a \in l_y$. For each $X \in \mathcal{U}$ such that $X \subset \mathbb{H}_y \cup \{a\}$ there exists $X' \in \mathcal{U}$ such that $X' \subset \mathbb{H}_{-y} \cup \{-a\}$, and the map $X \mapsto X'$ is injective.

Then y is a good direction.

Observe that Condition *(b)* holds for symmetric families (i.e. $X \in \mathcal{U} \implies -X \in \mathcal{U}$) because $X' = -X$ works.

Proof. In fact, fix $L \in \mathbb{N}$, and consider any configuration in $\eta \in \{+,-\}^{\varrho}$. We will show that

$$\sum_{v\in\eta^+} r_v(\eta) \leq \sum_{u\in\eta^-} r_u(\eta). \tag{10}$$

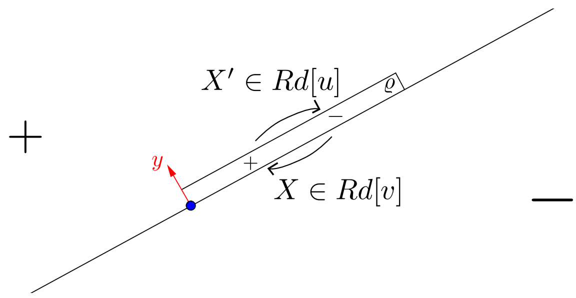

Figure 5: $\mathbb{H}_y$ entirely $-$ and $(\mathbb{H}_y \cup \varrho)^c$ entirely $+$

If $(v, X) \in \eta^+ \times \mathcal{U}$ is counted in LHS of (10), this is because $v \in \eta^+$ and X disagrees with v so $v + X$ is entirely $-$, moreover, since $y \in \mathcal{S}$, the set $v + X$ must have a vertex $u \in \eta^-$ and only 1 by *(a)*, so $v + X \subset \mathbb{H}_y \cup \{u\}$ or $X \subset \mathbb{H}_y \cup \{u - v\}$ so by *(b)* $\exists X' \in \mathcal{U}$ with $X' \subset \mathbb{H}_{-y} \cup \{v - u\}$ or $u + X' \subset \mathbb{H}_{-y} \cup \{v\}$, so $u + X'$ in entirely $+$ (see Figure 5) meaning that X' disagrees with u and $(u, X') \in \eta^- \times \mathcal{U}$ is counted in RHS of (10). Since the map $(v, X) \mapsto X'$ is an injection, for any pair (v, X) that contributes 1 in LHS we can find a contribution of 1 in RHS in a one to one way, so inequality (10) follows. ♦

Now, as we said in the introduction, by using this lemma and drawing the rules of the families $\mathcal{U}_a(\mathcal{U}', \mathcal{V}, \mathcal{W})$, it becomes evident that e_2 is a good direction for them. It looks like such families are the only examples we can give satisfying *(a)* and *(b)*. However we are free to construct a lot of different nature by using the following trivial observation:

Remark 3.10 (Adding rules)**.** Infinitely many critical families with a good direction can be constructed from a single critical family, just by properly adding new rules $X \subset \mathbb{H}_{-y}$, to maintain criticality, we just do it without changing the stable set (see Remark **??**).

3.3.1 Simulations

For a concrete example of critical families without good directions consider the collection of all subsets of size 3 of $\{e_1, -e_1, e_2, -e_2, 2e_1, -2e_1, 2e_2, -2e_2\}$, call it $\mathcal{U}_{3,8}$. It is easy to check that $\mathcal{S}(\mathcal{U}_{3,8}) = \{e_1, -e_1, e_2, -e_2\}$. To check that no direction in $\mathcal{S}(\mathcal{U}_{3,8})$ is good, by symmetry it is enough to consider $y = e_2$, and in the 1-dimensional setting, calculations show that configurations which alternate four $+$s and one $-$ do not satisfy (5). However simulations show that (see Figure **??**) we can erode the segment perpendicular to y in time $O(L^{2.1})$, which is a sufficient condition to be poly-eroding as we can read in the proof of Proposition 18.

here a heavy picture

Another such family, which is special since its droplets are triangular, is

$$\mathcal{U}_{\triangleright} = \{\{(-1,1),(-1,-1)\},\{(0,1),(1,1)\},\{(0,-1),(1,-1)\}\}$$

(see Figure 6), since $\mathcal{S}(\mathcal{U}_{\triangleright}) = \{(-1,0), \frac{1}{\sqrt{2}}(1,1), \frac{1}{\sqrt{2}}(1,-1)\}$ this family only has 3 candidates to be y and all of them fail Condition (5). In fact, say we choose $y = (-1,0)$ and

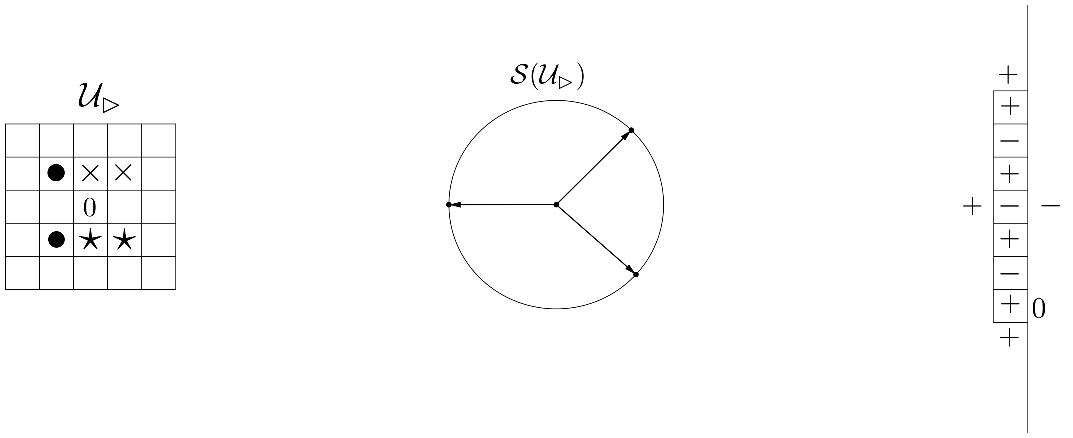

Figure 6: A non countervailed family, its stable set and some failing configurations

take $L = 2n+1$ for some fixed n. The configuration η where $\eta^+ = \{(0,2k) : k = 0, ..., n\}$ gives $\sum_{u\in\eta^-} Rul(u) = n$ while $\sum_{v\in\eta^+} Rul(v) = 2n$. Analogous situation for the other 2 candidates.

For this family, simulations show that (see Figure **??**) in fact we can erode the segment perpendicular to $y = (-1,0)$ in time $O(L^{3/5})$!.

here another heavy picture

To finish the discussion, we exemplify how to use Remark 24:

Remark 3.11. The direction $y = (-1,0)$ is good for the family $\mathcal{U}_{\triangleright} \cup \{\{(-1,2),(-1,-1)\}\}$, this is just because we added a rule in the $+$ side to compensate inequality (10) since the it has the same stable set as $\mathcal{U}_{\triangleright}$ then it is critical and poly-eroding so our main theorem holds for this new family. In general, if we consider any rule $X_0 \subset \mathbb{H}_{e_1}$, $X_0 \neq \{(-1,1),(-1,-1)\}$ such that there exist $x, x' \in X_0$ with $x_2 \geq -x_1$ and $x'_2 \leq x'_1$, we can check that the family $\mathcal{U}_{\triangleright} \cup \{X_0\}$ has $y = (-1,0)$ as a good direction and has the same stable set as $\mathcal{U}_{\triangleright}$, and of course, we can construct infinitely many such families in this way.

4 The process inside boxes

We move to the proof of Theorem 2.1. The strategy starts by constructing a rapidly decreasing sequence $\{q_k\}_k$ with $q_0 = q = 1-p$, and consider big space and time scales.

Define $l_0 = 1, t_0 = 0$,

$$q_k = \exp(-a/q_{k-1}), \qquad l_k = \left\lfloor \frac{1}{q_{k-1}} \right\rfloor^6, \qquad t_k = \left(\frac{1}{q_{k-1}} \right)^5 \tag{11}$$

$$L_k = l_0 \cdot \dots \cdot l_k, \qquad T_k = t_0 + \dots + t_k. \tag{12}$$

Consider $B_k = [L_k]^2$ and, at time T_k tile $\mathbb{Z}^2$ into copies of B_k in the obvious way. We couple the U-voter dynamics with a *block-dynamics* which is more 'generous' to $-$, defined as follows, for every k

- At time T_k every copy of B_k is monochromatic and the $\mathcal{U}$-voter dynamics afresh; until it arrives at time T_{k+1}.
- As t is close to T_{k+1}, if there exists some copy of B_k inside some copy of B_{k+1} which is in state $-$ then at time T_{k+1} we declare the state of B_{k+1} to be $-$. Otherwise we declare it to be $+$.

Let B'_k be the block with the same center as B_k but of sidelength $\frac{5}{3}L_k$. Heuristically what we will do is the following: First we focus just on the process $\mathcal{P}$ which we define as $\mathcal{U}$-voter dynamics run on the graph $\mathbb{Z}^2[B'_k]$ with $+$ boundary conditions and show that (Sec. 3.1, 3.2) the probability that there exists a vertex in B'_k in state $-$ at time T_k is small. Then, in Section 3.3 we prove that the probability that at this time the state of every vertex in B_k in the original dynamics differs from the process $\mathcal{P}$ is also small. Finally, in Section 3.4 we put the pieces all together and deduce Theorem 2.1.

Define $\hat{q}_k$ as the probability that at time T_k the block B_k is in the state $-$ in the block-dynamics. So in particular $\hat{q}_0 = q_0$.
We claim that $\hat{q}_k \leq q_k$ for q small enough and proceed by induction on k. It is true for $k = 0$. Assume it holds for k and let's prove it for $k+1$. Let's consider the evolution inside B'_{k+1} with $+$ boundary conditions, during the interval $[T_k, T_{k+1})$. Define the event

$$F_{k+1} := \{\exists - \in B_{k+1} \text{ as } t \text{ is close to } T_{k+1}\}. \tag{13}$$

In the next two sections we will prove that $\mathbb{P}_p(F_{k+1}) \leq q_{k+1}/2$.

4.1 Bootstrapping the vertices in state $-$

Next, in $\mathbb{Z}_n^2$ we declare the initially infected set A to be the vertex with spins $-$ (in our setting $n = l_{k+1}$, every vertex in $\mathbb{Z}_n^2$ represents a copy of $[L_k]^2$, and A is q_k-random). Then run the covering algorithm (at every step, spins $+$ becoming $-$) until we stop with a finite collection of droplets $\{D_1, \dots, D_z\}$, each one entirely $-$, and consider the event

$$E = \{\text{diam}(D_i) \leq \varepsilon^2 n^{1/6}, \text{ for all } i = 1, \dots, z\}. \tag{14}$$

We claim that $\mathbb{P}_p(E^c) \leq q_{k+1}/4$. In fact, if some D_i has diameter bigger than $\varepsilon^2 n^{1/6}$, then by Lemma 2.5 there exists a covered droplet D with $\varepsilon^2 n^{1/6} \leq \text{diam}(D) \leq 3\varepsilon^2 n^{1/6}$. If Z

denotes the number of such droplets when A is q_k-random then by Markov inequality we have

$$\mathbb{P}_p(E^c) \le \mathbb{E}_p[Z] \le \sum_{s=\varepsilon^2 n^{1/6}}^{3\varepsilon^2 n^{1/6}} n^3 \binom{s^2}{\varepsilon s} q_k^{\varepsilon s} \le \sum_{s=\varepsilon^2 n^{1/6}}^{3\varepsilon^2 n^{1/6}} n^3 \left(\frac{esq_k}{\varepsilon}\right)^{\varepsilon s} \le n^3 \sum_{s=\varepsilon^2 n^{1/6}}^{3\varepsilon^2 n^{1/6}} (3e\varepsilon)^{\varepsilon s}$$

$$\le Cn^3 \exp(-\varepsilon^2 n^{1/6}) \le \frac{\exp(-2an^{1/6})}{4},$$

(here we used $n^{1/6}q_k \le 1$ and picked $a < \varepsilon^2/2$), by Lemma 2.6, since the number of droplets in $\mathbb{Z}_n^2$ with diameter s is $O(n^{2+1/6})$, and each has area at most s^2.
This gives $\mathbb{P}_p(E^c) \le q_{k+1}/4$, and since $\mathbb{P}(F_{k+1}) \le \mathbb{P}_p(F_{k+1}|E) + \mathbb{P}_p(E^c)$, then all we need to check now is that for a small enough we also have

$$\mathbb{P}_p(F_{k+1}|E) \le q_{k+1}/4 \tag{15}$$

This is the content of the next section.

4.2 Erosion step

To get (15) we use Proposition **??**, but then we need an upper bound of L_k as a function of q_k.

Lemma 4.1. *If q is small enough then the sequence $\{q_k\}_k$ is decreasing and for arbitrary $\delta > 0$,*

$$L_k \le (1/q_k)^{\delta}, \tag{16}$$

uniformly in k.

Proof. See equation (4.8) from [6]. ♦

By the Markov property we need to estimate the probability that starting at time T_k from a configuration in B_{k+1} where E holds and letting the system evolve with $+$ boundary conditions, some spin $-$ will be present as t is close to T_{k+1}. By attractiveness, an upper bound is obtained by starting the evolution at time T_k with $-$ spins at all sites of the droplets $D_1, \dots, D_z$ participating in E.
If E occurs, by (16), for small q each droplet has diameter at most $\varepsilon^2 l_{k+1}^{1/6} \cdot L_k \le \varepsilon^2 \frac{1}{q_k}\left(\frac{1}{q_k}\right)^{\delta} \le \left(\frac{1}{q_k}\right)^{1+\delta}$, so by Proposition **??**, if q is small enough, for each $i = 1, \dots, z$, the probability that at time $T_{k+1} = T_k + (1/q_k)^5$ there is any spin $-$ inside D_i is at most

$$\mathbb{P}_p\left[T > \left(\frac{1}{q_k}\right)^5\right] \le \mathbb{P}_p\left[T > \gamma\left(\frac{1}{q_k}\right)^{4(1+\delta)}\right] \le \exp\{-\frac{1}{2}\left(\frac{1}{q_k}\right)^{1+\delta}\}.$$

For small q we also have $z \le |B'_{k+1}| \le [(5/3)L_{k+1}]^2 \le \frac{1}{q_{k+1}}$, by (16) again. Therefore

$$\mathbb{P}_p[F_{k+1}|E] \le \frac{1}{q_{k+1}} \exp\{-\frac{1}{2}\left(\frac{1}{q_k}\right)^{1+\delta}\} \le \frac{q_{k+1}}{4},$$

for small q, and this concludes the proof of $\mathbb{P}_p(F_{k+1}) \le q_{k+1}/2$.

5 Wrapping up

Here we argue that the probability of having some $-$ inside B_{k+1} with the help of some vertex outside B'_{k+1} on time $[T_k, T_{k+1})$ is also small, and then finish the proof.

5.1 Control of the outer influence

We call a sequence $(x_1, s_1), \ldots, (x_r, s_r)$ of vertex-time pairs, where $x_i \in \mathbb{Z}^2$ and $s_i \geq 0$, a *path of clock rings* (and say that such a sequence is a path from x_1 to x_r in time $[s_1, s_r]$) if

1. $0 < \|x_{i+1} - x_i\|_1 \leq C$ for each $i \in [r-1]$ (C constant).
2. $s_1 < \cdots < s_r$.
3. The clock of vertex x_i rings at time s_i for each $i \in [r]$.

The key point is that if there does not exist a path of clock-rings from x_1 to x_r in time $[s_1, s_r]$, then the state of vertex x_r at time s_r is independent of the state of vertex x_1 at time s_1.

Now consider the event F'_{k+1} that there exists a path of clock-rings from some vertex outside B'_{k+1} to some vertex inside B_{k+1} in time $[T_k, T_{k+1}]$. So, if F'_{k+1} does not occur, then the state of every vertex in B_{k+1} at time T_{k+1} is the same in the $\mathcal{U}$-voter dynamics as it is in the process $\mathcal{P}$, since the boundary conditions cannot affect B_{k+1}. This gives $\hat{q}_{k+1} \leq \mathbb{P}_p(F_{k+1}) + \mathbb{P}_p(F'_{k+1})$, we will prove

$$\mathbb{P}_p(F'_{k+1}) \leq q_{k+1}/2, \tag{17}$$

concluding our induction step.

In fact, on F'_{k+1} every such a path have length at least $\pi_k \geq \left\lfloor \frac{1}{3C} L_{k+1} \right\rfloor \geq \frac{1}{6C} \left(\frac{1}{q_k} \right)^6$. Now by (16), for each $r \in \mathbb{N}$ there exist at most $(C' L_{k+1})(4C^2)^r \leq \frac{C'}{q_{k+1}} (4C^2)^r$ paths of length r starting on $\partial B'_{k+1}$ (C' constant). Denote $P_k(r)$ the probability that there exist times $T_k \leq s_1 < \cdots < s_r \leq T_{k+1}$ such that $(x_1, s_1), \ldots, (x_r, s_r)$ is a path of clock rings. Observe that $P_k(r)$ does not depend on the choice of the path. We need to bound $P_k(r)$: For every $j \in [r]$ choose s_j to be the first time the clock at x_j rings after time s_{j-1}. Denote G_m the event that $s_m - s_{m-1} \leq 2t_{k+1}/r$, so $\mathbb{P}_p(G_m) = 1 - \exp(2t_{k+1}/r) \leq 2t_{k+1}/r$, and the events G_m are independent, therefore

$$P_k(r) = \mathbb{P}_p(s_r \leq t_{k+1}) \leq \mathbb{P}_p \left(\sum_{m=1}^{r} \mathbb{1}_{G_m} \geq r/2 \right) \leq \binom{r}{r/2} \left(\frac{2t_{k+1}}{r} \right)^{r/2} \leq \left(\frac{8t_{k+1}}{r} \right)^{r/2}.$$

Finally observe that for $r \geq \pi_k$ we have $r/t_{k+1} \geq \frac{1}{6C}\frac{1}{q_k}$ so

$$\mathbb{P}_p(F'_{k+1}) \leq \sum_{r=\pi_k}^{\infty} \frac{C'}{q_{k+1}}(4C^2)^r \left(\frac{8t_{k+1}}{r}\right)^{r/2} \leq \frac{C'}{q_{k+1}} \sum_{r=\pi_k}^{\infty} (C''q_k)^{r/2}$$
$$\leq \frac{C'}{q_{k+1}} \exp\left[-\hat{C}\left(\frac{1}{q_k}\right)^6\right] \leq q_{k+1}/2.$$

5.2 All together now

Set $\beta = 1/5$. First we observe that Theorem 2.1 holds for all times T_k, because

$$\mathbb{P}_p(\sigma_{T_k}(0) = -) \leq \hat{q}_k \leq q_k = \exp(-at_k^\beta).$$

But $t_{k-1}/t_k = (q_{k-1}/q_{k-2})^5 \leq c$ for some constant $c < 1$, so $T_k \leq (1-c)^{-1}t_k$ and

$$\mathbb{P}_p(\sigma_{T_k}(0) = -) \leq \exp(-\hat{C}T_k^\beta).$$

Therefore, Theorem 2.1 holds for all times of the form $t = T_k$. To conclude it holds for all $t > 0$, we use a coupling trick which appears in [6] which consists of comparing evolutions started from product measures with different values of q. Let's rewrite q_k, t_k, T_k as $q_k(q), t_k(q), T_k(q)$ because they depend on the initial q. We have shown that there exists some $a' > 0$, such that for every $0 < q \leq a'$, when $t = T_k(q)$ it holds that

$$\mathbb{P}_p(\sigma_t(0) = -) \leq \exp(-\hat{C}t^\beta). \tag{18}$$

Now write $q'_k = q_k(a'), t'_k = t_k(a')$ and $T'_k = T_k(a')$, and observe that q'_k decreases (so t'_k increases) as k increases. For fixed k, consider the parameter q decreasing continuously from a' to q'_1, so the corresponding $T_k(q)$ increasing continuously from $T_k(a')$ to $T_k(q'_1) = t_1(q'_1)+\cdots+t_k(q'_1) = T'_{k+1} - t'_1$. By continuity of $T_k(q)$ and the intermediate value theorem, any $t \notin \cup_{k\geq 1}[T'_k - t'_1, T'_k)$ can be written as $t = T_{k(t)}(q[t])$, for some $k(t) \geq 1$, $q'_1 < q[t] \leq a'$. Observe that $p = 1 - q \geq 1 - q'_1 \geq 1 - q[t] =: p[t]$. Combining monotonicity and (18) we get $\mathbb{P}_p(\sigma_t(0) = -) \leq \mathbb{P}_{p[t]}(\sigma_t(0) = -) \leq \exp(-\hat{C}t^\beta)$, in summary, now we have shown the theorem holds for all $q < q'_1$ and $t \notin \cup_{k\geq 1}[T'_k - t'_1, T'_k)$.
Finally, suppose $t \in [T'_k - t'_1, T'_k)$ for some k. The clue observation is, if $\sigma_t(0) = -$ and the spin at the origin does not flip between times t and T'_k then $\sigma_{T'_k}(0) = -$.

$$\mathbb{P}_p[\sigma_t(0) = -] \leq e^{t'_1}\mathbb{P}_p[\sigma_{T'_k}(0) = -] \leq e^{t'_1}\exp(-\hat{C}{T'_k}^\beta) \leq C_1\exp(-C_2t^\beta),$$

since the probability that no flips occur at the origin from t to T'_k is at least $e^{-t'_1}$. ♦

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