HOMEWORK #1

[1] Solve all exercises for Sections §1.2 and §1.3

[2] Let A, B, C be events. Match each set equation to the left with its corresponding Probability Jargon to the right:

(a) $A \cap B \cap C = A \cup B \cup C$	(i) A and "B or C" are disjoint
(b) $A \cap B \cap C = A$	(ii) The events A, B, C are the same
(c) $A \cup B \cup C = A$	(iii) If A occurs then "B and C" occur
(d) $(A \cup B \cup C) \setminus (B \cup C) = A$	(iv) If "B or C" occurs then A occurs.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space: ($A \in \mathcal{F}$ means $A \subseteq \Omega$ is an event)

[3] Prove the UNION BOUND:

$$A_1, A_2, \ldots \in \mathcal{F} \implies \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_n\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

If moreover, $\mathbb{P}(A_i) = 0, \ \forall i \in \mathbb{N} \implies \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$.

[4] Let $A_1, A_2, \ldots \in \mathcal{F}$ and $B_1, B_2, \ldots \in \mathcal{F}$ such that $\mathbb{P}(A_n) \to 1$ and $\mathbb{P}(B_n) \to p$, as $n \to \infty$. Prove that $\mathbb{P}(A_n \cap B_n) \xrightarrow[n\to\infty]{} p$.

[5] Let $A_1, A_2, \ldots \in \mathcal{F}$ and define:

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad \text{and} \quad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

Assume that $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$ and call this event B.

Prove that $\mathbb{P}(A_n) \longrightarrow \mathbb{P}(B)$, as $n \to \infty$.

Optional! Let Ω be a non-empty set.

(a) If $\{\mathcal{F}_i : i \in I\}$ is a collection of σ-fields on Ω, prove that $\bigcap_{i \in I} \mathcal{F}_i$ is a σ-field on Ω. (I could be uncountable!)

(b) Fix any arbitrary $\mathcal{A} \subseteq 2^{\Omega}$. Show that there is a σ-field $\mathcal{F} \supseteq \mathcal{A}$ such that any other σ-field $\mathcal{F}_2 \supseteq \mathcal{A}$ must satisfy $\mathcal{F} \subseteq \mathcal{F}_2$. $\mathcal{F}$ is called the "smallest" σ-field containing $\mathcal{A}$.

WORKED EXAMPLES

Example 1: If 2 dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

Solution: Since there are 2 dice, the sample space is $\Omega = \{(i,k) : i,k \in \{1,2,3,4,5,6\}\}$ and $|\Omega| = 36$.
Assuming that the dice are fair, all 36 possible outcomes are equally likely. Thus

$$\mathbb{P}((i,k)) = \frac{1}{|\Omega|} = \frac{1}{36}, \quad \forall (i,k) \in \Omega.$$

The event of interest corresponds to the following set:

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

(this is the set of pairs that sum up to 7). Thus

$$\mathbb{P}(A) = \mathbb{P}((1,6)) + \mathbb{P}((2,5)) + \mathbb{P}((3,4)) + \mathbb{P}((4,3)) + \mathbb{P}((5,2)) + \mathbb{P}((6,1)) = 1/6.$$

Example 2: A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Solution: The sample space consists of all distinct groups of 5 people that can be chosen from $6+9=15$ people, therefore $|\Omega| = \binom{15}{5}$
"Randomly selected" means that each of the $\binom{15}{5}$ possible combinations is equally likely to be selected.
There are $\binom{6}{3}\binom{9}{2}$ ways to choose 3 men and 2 women, so

$$\mathbb{P}(3 \text{ men}, 2 \text{ women}) = \binom{6}{3}\binom{9}{2} \Big/ \binom{15}{5} = \frac{240}{1001}.$$

Example 3: If n people are present in a room, what is the probability that no 2 of them celebrate their birthday on the same day of the year? How large need n be so that this probability is $\leq 1/2$?

Solution: $\Omega = \{(w_1, \dots, w_n) : w_i \in \{1, 2, \dots, 365\} \ \forall i = 1, \dots, n\}$

w_i represents the birthday of the i-th person, and we are ignoring that someone has been born on February 29.

So, $|\Omega| = (365)^n$. There are $(365)(364)(363) \cdot \ldots \cdot (365-n+1)$ ways of choosing n people with different birthdays, so the desired probability is $(365)(364)\cdots(365-n+1) \Big/ (365)^n$

Moreover, it is less than $1/2$ if $n \geq 23$. $\boxed{\text{CHECK IT!}}$

That means, if there are 23 or more people in a room, then it is very likely that 2 of them have same birthday.

Applied Problems

$\boxed{6}$ A die is rolled continually until a 6 appears then we stop the experiment. What is a suitable sample space?

Let $E_n = \{n \text{ rolls are necessary to complete the experiment}\}$

What points of Ω are in E_n? What is $\left(\bigcup_{n=1}^{\infty} E_n\right)^c$?

$\boxed{7}$ If 8 rooks are randomly placed on a chessboard, compute the probability that none of the rooks can capture any of the others. The event is:

$A = \{\text{no row or column contains more than 1 rook}\}$.

$\boxed{8}$ A group of individuals containing b boys and g girls is lined up in random order — i.e: each of the $(b+g)!$ permutations is equally likely — what is the probability that the person in the i-th position with $1 \leq i \leq b+g$, is a girl?

$\boxed{9}$ Two dice are thrown n times in sucesion. Compute the probability that $(6,6)$ appears at least once. How large need n be to make this probability at least $1/2$?

$\boxed{10}$ If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.