HOMEWORK #2

1 Section 1.4: Exercises 1, 2, 3, 5(a)(i), 7.

2 Section 1.5: Exercises 1, 2, 3, 4, 7, 9.

3 Let $B, C, A_1, A_2, A_3, \dots$ be events. Prove that

(a) If $A_1, A_2, \dots$ are pairwise disjoint, $\mathbb{P}(A_n) > 0$ and $\mathbb{P}(B|A_n) \geq p$ for all $n \in \mathbb{N}$, then $\mathbb{P}(B | \bigcup A_n) \geq p$.

(b) If $A_1, A_2, \dots$ are decreasing and $\mathbb{P}(A_{n+1}|A_n) \leq 1/2$, $\forall n \in \mathbb{N}$, then $\mathbb{P}(A_n) \to 0$, as $n \to \infty$.

(c) If $A_1, A_2, \dots$ are pairwise disjoint and $\mathbb{P}(B|A_n) = \mathbb{P}(C|A_n)$, $\forall n$ then $\mathbb{P}(B | \bigcup_n A_n) = \mathbb{P}(C | \bigcup_n A_n)$.

(d) If $A_1, A_2, \dots$ is a partition of Ω, then $\mathbb{P}(B|C) = \sum_n \mathbb{P}(A_n|C)\mathbb{P}(B|A_n \cap C)$

4 (Multiplication Property) If $A_1, \dots, A_n \in \mathcal{F}$, then

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2|A_1)\,\mathbb{P}(A_3|A_1 \cap A_2) \times \dots \times \mathbb{P}(A_n|A_1 \cap \dots \cap A_{n-1})$$

5 (Bayes' Formula) Let $B, A_1, A_2, \dots$ be events with positive probability such that $A_1, A_2, \dots$ is partition of Ω. Then

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\sum_n \mathbb{P}(A_n)\,\mathbb{P}(B|A_n)}$$

WORKED EXAMPLE: Suppose that a box contains 3 coins: 2 normal coins and 1 fake coin (meaning, with two heads). Randomly pick one coin and toss it. What is the probability of that coin being the fake one, given that the result was a head?

Sol: Consider the events

	Normal 1	Normal 2	Fake
Front →	H	H	H
Back →	T	T	H

$B = \{\text{the result was H}\}$

$A_1 = \{\text{the picked coin was normal}\}$

$A_2 = A_1^c = \{\text{the picked coin was the fake one}\}$

A_1, A_2 is a partition of Ω, so by Bayes' formula:

$$\mathbb{P}(A_2|B) = \frac{\mathbb{P}(A_2)\mathbb{P}(B|A_2)}{\mathbb{P}(A_1)\mathbb{P}(B|A_1)+\mathbb{P}(A_2)\mathbb{P}(B|A_2)} = \frac{\frac{1}{3}\times 1}{\frac{2}{3}\times\frac{1}{2}+\frac{1}{3}\times 1} = \frac{1}{2}$$

[6] Toss 2 fair dice. Given that the result shows different numbers, what is the conditional prob. that:

(a) at least one die shows 6.

(b) the sum of the numbers is 8.

[7] In a multiple choice test, with $\underline{\underline{m}}$ choices, the probability of a student knowing the answer is $\underline{\underline{p}}$. If she knows the answer then she chooses the right answer with prob. 1. But, if she doesn't then she chooses the right answer with prob. $1/m$. What is the prob. that she knew the answer given that she chose the right answer? Compute the limit of this prob. as

(i) $m \to \infty$, p fixed. (ii) $p \to 0$, m fixed.

[8] Let $A_1, \dots, A_n \in \mathcal{F}$ be independent and $p_k = \mathbb{P}(A_k)$ for $k = 1, \dots, n$. Obtain the prob. of the following events in terms of $p_1, \dots, p_n$:

(a) The occurrence of none of the A_k's.

(b) " " of at least one A_k.

(c) " " of exactly one A_k.

(d) " " of exactly two A_k's.

(e) " " of all A_k's.

(f) " " of <u>at most</u> $n-1$ of the A_k's.