Math 145, Winter 2020
January 24, 2020

MIDTERM EXAM 1 (sample)

NAME(use CAPITAL letters, *first name first*): Full name

NAME(sign): Signature

ID#:______________________

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. You can use the Formula Sheet provided at our course website. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

Before starting: Imagine that this is your exam and solve it in 50 minutes.

1. There are more four-letter sequences that can be formed by using the letters in Bobo, Mississippi, than can be formed by using the letters in Soso, Mississippi. Is the difference more or less than the distance between these two cities in miles, which is 267?

The possible patterns are wwww, wwwx, wwxy, wwxx, and wxyz; and for each such pattern the number of strings using SOSO does not exceed the number of strings using BOBO.

Now, just by counting the patterns wwxy, we get that the difference is at least

$$\binom{5}{1}\binom{5}{2}\times\binom{4}{2,1,1} - \binom{4}{1}\binom{4}{2}\times\binom{4}{2,1,1}$$

why?

$$= (50-24)\times 12 = 312$$

$$> 267.$$

2. Compute the value of the following sum in terms of an expression involving only one binomial coefficient: For $m, n \in \mathbb{N}$,

$$\sum_{k \le m} (-1)^k \binom{n}{k}.$$

Call this sum $S_{m,n}$. Let's compute some values:

Recall that $\binom{n}{k} = 0$ for $k<0$

$S_{1,1} = (-1)^0\binom{1}{0} + (-1)\binom{1}{1} = 1-1=0$

$S_{1,2} = \binom{2}{0} - \binom{2}{1} = 1-2 = -1$

$S_{1,3} = \binom{3}{0} - \binom{3}{1} = 1-3 = -2$, and for general n:

$S_{1,n} = \binom{n}{0} - \binom{n}{1} = 1-n = (-1)\binom{n-1}{1}$

Now, for $m=2$ and general n:

$S_{2,n} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} = -\binom{n-1}{1} + \binom{n}{2} = \binom{n-1}{2}$

Previous step Addition property

Thus, our natural guess is $\boxed{S_{m,n} = (-1)^m \binom{n-1}{m}}$

We can easily prove it by induction on m.

Do it!

3. [20pts]. How many numbers from 1 to 100 are a multiple of at least one of the numbers 3, 5 and 7?

Recall that given any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x, i.e.

4. [20 pts]. Let F_n be the n-th Fibonacci number, i.e. $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that for $m, n \in \mathbb{N}$,

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n.$$

5. Determine whether the following statements are True or False. Justify your answer with a proof or a counterexample as appropriate.

(a) [10pts]. Each arrangement of the integers between 1 and 9 has either an increasing or decreasing subsequence of length 4.

(b) [10pts]. Given five points inside an equilateral triangle of side length 2, there exist two points whose distance from each other is at most 1.