Math 135A, Fall 2019
October 21, 2019

MIDTERM EXAM 1 (sample)

NAME(use CAPITAL letters, *first name first*):____

NAME(sign):________________________________

ID#:_______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. You can use the Formula Sheet provided at our course website. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

Before starting: Imagine that this is your exam and solve it in 50 minutes. Problems 1 and 3 come with solutions, so solve them on a different paper. **Avoid reading the solutions**, and compare them with yours only when you finish your simulation.

1. Eight fair dice are rolled.

(a) Compute the probability that all numbers rolled are the same.

$$6/6^8 = \frac{1}{6^7}$$

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(b) Compute the probability that each of the four numbers 1, 2, 3, 4 is rolled exactly twice.

$$\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{6^8} = \frac{8!}{2^4 \cdot 6^8}$$

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(c) Compute the probability that exactly two numbers are represented among the numbers rolled (i.e., the set of numbers rolled contains exactly two elements).

choose the two numbers, which occur on each roll, but not only one of them in all rolls

$$\frac{1}{6^8} \cdot \binom{6}{2}\left(2^8 - 2\right)$$

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(d) Compute the probability that each of the four numbers 1, 2, 3, 4 is represented among the numbers rolled.

$A_i = \{\text{number } i \text{ missing}\}$

$P(A_1) = \left(\frac{5}{6}\right)^8$, $P(A_1 \cap A_2) = \left(\frac{4}{6}\right)^8$ 10

$P(A_1 \cap A_2 \cap A_3) = \left(\frac{3}{6}\right)^8$, $P(A_1 \cap A_2 \cap A_3 \cap A_4) = \left(\frac{2}{6}\right)^8$

Answer. $1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$

$$= 1 - 4\cdot\left(\frac{5}{6}\right)^8 + \binom{4}{2}\left(\frac{4}{6}\right)^8 - \binom{4}{3}\left(\frac{3}{6}\right)^8 + \left(\frac{2}{6}\right)^8.$$

2. Four numbers $n_1 < n_2 < n_3 < n_4$ are randomly distributed to players $1, 2, 3, 4$. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers, the winner then compares with player 3, and finally, the new winner compares with player 4. Let X denote the number of times player 1 is a winner.

(a) [10pts] Find $\mathbb{P}(X = k)$ for $k = 0, 1, 2, 3$.

(b) [7pts] Compute $\mathbb{E}(X)$.

(c) [8pts] Plot the distribution function of X.

3. Shuffle a full deck of 52 cards.

(a) What is the probability that the top three cards are an Ace, a King and a Queen, in any order?

$$\frac{4^3}{\binom{52}{3}}$$

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(b) What is the probability that the top three cards are of three different suits?

choose suits, then cards from chosen suits

$$\frac{\binom{4}{3} \cdot 13^3}{\binom{52}{3}}$$

8

(c) Toss ~~four~~ three fair coins, note the number N of Heads and examine top N cards. (For example, if your tosses are all Tails, you examine no cards at all; if you toss 2 Heads and 1 Tails, you examine top 2 cards.) Compute the probability that there is at least one Ace among the examined cards.

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$F_i = \{\text{toss } i \text{ H's}\}$

$A = \{\text{at least one A}\}$

$$P(A) = \sum_{i=0}^{3} P(F_i)\, P(A|F_i)$$

$$= \frac{1}{2^3} \cdot 0 + \frac{\binom{3}{1}}{2^3} \cdot \frac{1}{13} + \frac{\binom{3}{2}}{2^3} \cdot \left(1 - \frac{48}{52} \cdot \frac{47}{51}\right)$$

$$+ \frac{1}{2^3}\left(1 - \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50}\right)$$

4. Determine whether the following statements are True or False. Justify your answer with a proof or a counterexample as appropriate.

(a) [7pts]. If two events are disjoint, then they are independent.

(b) [6pts]. If X is a random variable and $\mathbb{P}(X < 7) = 1$, then $\mathbb{P}(X > 9) = 0$.

(c) [6pts]. If $X \sim \text{Ber}(p)$ and $Y = 1 - X$, then $Y \sim \text{Ber}(p)$.

(d) [6pts]. If $X \sim \text{Geo}(p)$, then $\mathbb{P}(X = k + i | X > i) = \mathbb{P}(X = k)$ for all $i, k \in \mathbb{N}$.