Math 108, SS1 2020
July 23, 2020

**FINAL EXAM (Sample)**

NAME(use CAPITAL letters, *first name first*): NAME

NAME(sign): Sign

ID#: 123. .

**HONOR STATEMENT**: By signing this paper, I hereby declare that I solved this exam by my own, without any external collaboration (like friends, internet solutions, etc). If needed, I am allowed to use our lecture notes only. I understand that the main purpose of this exam is to show how much I have learned in this course, holding myself to a high standard of academic integrity, and that suspected misconduct on this exam will be reported to the Office of Student Support and Judicial Affairs and, if established, will result in disciplinary sanctions up through Dismissal from the University and a grade penalty up to a grade of 'F' for the course.

**Instructions**: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided, or answer it in a separate paper and sign that paper; it is optional to print this exam. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Unless directed to do so, do *not* prove any theorem or proposition seen in class, and do not evaluate complicated expressions to give the result as a fraction or a decimal number. However, if you are using any of the problems in the textbook, then you have to solve or prove it.

**To deliver**: Submit your solutions on Canvas, as you did with the Review Hw, the due time is 3:00PM. Sign and submit the honor statement (write it down by hand, if needed). Submit your self-video on Canvas as well, before 6:00PM.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

| | |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| TOTAL | Sum/200 |

*Remark*: In this exam, $\mathbb{N} = \{1, 2, 3, \dots\}$, so that $0 \notin \mathbb{N}$.

Problem 1.

Prove the following two statements. You may use (a) to prove (b).

(a) The number $\sqrt{3}$ is irrational.

Assume $\sqrt{3} \in \mathbb{Q}$; then $\sqrt{3} = \frac{p}{q}$, where $p$ and $q$ are natural numbers without a common factor, then $3 = \frac{p^2}{q^2}$, $p^2 = 3q^2$ and $3 \mid p^2$. Thus $3 \mid p$ and $p = 3k$ for some $k$, but then $9k^2 = 3q^2$, $q^2 = 3k^2$, $3 \mid q^2$ and $3 \mid q$. So $p$ and $q$ have a common factor 3, a contradiction.

(b) The number $\frac{1}{2}\sqrt{3} + 5$ is irrational.

Assume $\frac{1}{2}\sqrt{3} + 5 \in \mathbb{Q}$. Then $\frac{1}{2}\sqrt{3} + 5 = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$, and then
$$\sqrt{3} = \left(\frac{p}{q} - 5\right) \cdot 2 = \frac{2p - 10q}{q}$$

As $2p - 10q \in \mathbb{Z}$ and $q \in \mathbb{Z}$, $\sqrt{3} \in \mathbb{Q}$, a contradiction.

*Problem 2,* [redacted]

(a) Give a counterexample: $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

$A = \{1\}$, $B = \{2\}$. Then $\{1,2\} = A \cup B \in \mathcal{P}(A \cup B)$, but $\{1,2\} \not\subseteq \{1\}$ and $\{1,2\} \not\subseteq \{2\}$, so $\{1,2\} \notin \mathcal{P}(A)$ and $\{1,2\} \notin \mathcal{P}(B)$, and then $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

(b) Prove: If $A$ is denumerable and $B$ is finite, then $A - B$ is denumerable.

$A - B$ is <u>countable</u>, as a subset of countable set $A$, i.e., $A - B \subseteq A$.

$A - B$ is <u>infinite</u>; if it were finite, then $A = B \cup (A-B)$ would be a union of two finite sets, thus finite. Thus, $A-B$ is denumerable, as infinite countable set.

(c) Give a counterexample: If $A$ is denumerable and $B$ is denumerable, then $A \cap B$ is denumerable.

Take $A = \{2, 4, 6, 8, \dots\} = \{2k : k \in \mathbb{N}\}$
$B = \{1, 3, 5, \dots\} = \{2k-1, k \in \mathbb{N}\}$

Then $A$ and $B$ are denumerable, but $A \cap B = \emptyset$, thus finite.

## Problem 3

Assume $A$, $B$, and $C$ are arbitrary sets and $f : A \to B$ and $g : B \to C$ are arbitrary functions. Assume also that $D$ and $E$ are arbitrary subset of $A$. For each statement below, prove it or provide a counterexample.

(a) Prove: If $f(D) \cap f(E) = \emptyset$ then $D \cap E = \emptyset$.

Contrapositive:
Assume $D \cap E \neq \emptyset$. Then $\exists x \in D \cap E$, but then $f(x) \in f(D)$ and $f(x) \in f(E)$, and so $f(D) \cap f(E) \neq \emptyset$.

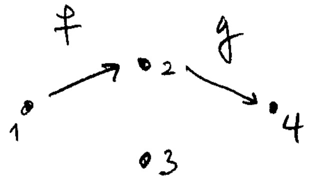
(b) Prove: If $f$ is one-to-one and $D \cap E = \emptyset$, then $f(D) \cap f(E) = \emptyset$.

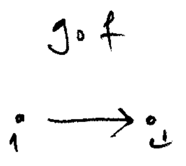
Assume $f$ is one-to-one, and that $f(D) \cap f(E) \neq \emptyset$. Then there $\exists y \in f(D) \cap f(E)$. Then $y = f(x_1)$ for some $x_1 \in D$ and $y = f(x_2)$ for some $x_2 \in E$. But $f$ is one-to-one, so $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Thus $x_1 \in D$ and $x_1 = x_2 \in E$, $x_1 \in D \cap E$. So $D \cap E \neq \emptyset$.

(c) Prove: If $f$ and $g$ are both onto, then $g \circ f$ is onto.

Take a $z \in C$. Then there exists a $y \in B$ so that $g(y) = z$ and then an $x \in A$ so that $f(x) = y$. Thus $g(f(x)) = z$, i.e. $(g \circ f)(x) = z$.

(d) Give a counterexample: If $g \circ f$ is onto, then $f$ is onto.




f is not onto

g∘f is onto

Problem 4

Determine the cardinality of each of the following sets. Give the result as either 0, a natural number, $\aleph_0$, or $\mathbf{c}$.

(a) $\{1,2,3,4\}^{\{1,2\}}$

$4^2 = 16$

(b) $\{1,2,3\} \times \{1,2\}$

$3 \cdot 2 = 6$

(c) $\mathcal{P}(\{1,2,3\} \times \{1,2\})$

$2^6 = 64$

(d) $\{\frac{7}{2^n} : n \in \mathbb{N}\}$ $\aleph_0$, as this is an infinite subset of $\mathbb{Q}$ (which is denumerable).

(e) $[2,3]$ $c$ as $(0,1) \approx (2,3) \subseteq [2,3] \Rightarrow \overline{\overline{[2,3]}} \geq c$

and $[2,3] \subseteq \mathbb{R} \approx (0,1) \Rightarrow \overline{\overline{[2,3]}} \leq c$

Cantor-Bernstein Thm. implies $\overline{\overline{[2,3]}} = c$.

(f) $\mathbb{Q} \times \mathbb{N}$ $\approx \mathbb{N} \times \mathbb{N} \approx \mathbb{N}$

$\aleph_0$

(g) $\mathbb{R} \times \mathbb{N}$ $c$

$\mathbb{R} \times \mathbb{N} \subseteq \mathbb{R} \times \mathbb{R} \approx \mathbb{R}$, thus $\overline{\overline{\mathbb{R} \times \mathbb{N}}} \leq c$.

$\mathbb{R} \approx \mathbb{R} \times \{1\} \subseteq \mathbb{R} \times \mathbb{N}$, thus $\overline{\overline{\mathbb{R} \times \mathbb{N}}} \geq c$.

By Cantor-Bernstein, $\overline{\overline{\mathbb{R} \times \mathbb{N}}} = c$.

Problem 5. Determine whether the following statements are True or False. Justify your answer with a proof or a counterexample as appropriate.

(i) If $H$ and $K$ are subgroups of a group $G$, then $H \cup K$ is a subgroup of $G$.

(ii) If $G$ is a group, then $Z(G) = \{g \in G : xg = gx \text{ for every } x \in G\}$ is an abelian subgroup of $G$.

(iii) $S_3$ is isomorphic to $(\mathbb{Z}_6, +)$.