Math 108, SS1 2020
July 31, 2020

FINAL EXAM

NAME(use CAPITAL letters, *first name first*): FULL NAME

NAME(sign):

ID#: 123...

HONOR STATEMENT: By signing this paper, I hereby declare that I solved this exam by my own, without any external collaboration (like friends, internet solutions, etc). If needed, I am allowed to use our lecture notes only. I understand that the main purpose of this exam is to show how much I have learned in this course, holding myself to a high standard of academic integrity, and that suspected misconduct on this exam will be reported to the Office of Student Support and Judicial Affairs and, if established, will result in disciplinary sanctions up through Dismissal from the University and a grade penalty up to a grade of 'F' for the course.

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided, or answer it in a separate paper and sign that paper; it is optional to print this exam. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Unless directed to do so, do *not* prove any theorem or proposition seen in class, and do not evaluate complicated expressions to give the result as a fraction or a decimal number. However, if you are using any of the problems in the textbook, then you have to solve or prove it.

To deliver: Submit your solutions on Canvas, as you did with the Review Hw. Sign and submit the honor statement (write it down by hand, if needed).
Submit your Video on Canvas as well, the due time is 10:00PM.
Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

Remark: In this exam, $\mathbb{N} = \{1, 2, 3, \dots\}$, so that $0 \notin \mathbb{N}$.

1. Show that, for every $n \geq 1$:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \quad (\Delta)$$

Proof: (i) For $n=1$. $LHS = \sum_{k=1}^{1} k^2 = 1 = \frac{1 \times 2 \times 3}{6} = RHS$.

(ii) Assume that (Δ) holds for n, and let's show it holds for $n+1$. In fact,

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \text{ by I.H.}$$

$$= (n+1)\left[\frac{n(2n+1)}{6} + n+1\right] = (n+1)\left[\frac{2n^2+n+6n+6}{6}\right]$$

$$= (n+1)\left[\frac{(n+2)(2n+3)}{6}\right] = \frac{(n+1)([n+1]+1)(2[n+1]+1)}{6},$$

So, (Δ) holds for $n+1$.

By (i), (ii) and P.M.I, (Δ) holds $\forall n \in \mathbb{N}$.

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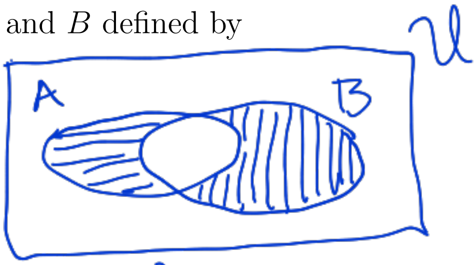
2. Let U be a set, and let $A \subseteq U$. Recall the *indicator function* $\chi_A : U \to \mathbb{Z}_2$ defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Now, let $A, B \subseteq U$ and consider the *symmetric difference of* A and B defined by

$$A \triangle B = (A - B) \cup (B - A).$$



(a) Show that $A \triangle B \subseteq U$, and compute $\varnothing \triangle A$.

• $A \subseteq U \Rightarrow A - B = A \cap B^c \subseteq U$. Also, $B \subseteq U \Rightarrow B - A = B \cap A^c \subseteq U$. Thus $A \triangle B = (A-B) \cup (B-A) \subseteq U$.

(b) Prove that $\forall x \in U,\ \chi_{A \triangle B}(x) = \chi_A(x) + \chi_B(x)$, where addition is taken modulo 2 (so that $1 + 1 = 0$).

3. Show the following statements.

   (a) The set $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 < 1\}$ is uncountable.

   (b) $\overline{\overline{\mathbb{Q} \cap (0, \infty)}} \leq \overline{\overline{\mathbb{R} \times \mathbb{R}}}$

   (c) The interval $(0, 1)$ is equivalent to the interval $(1, 2]$.

4. Let $\mathcal{M}$ be the set of 2×2 matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

where $a, d \in \mathbb{R} - \{0\}$. Consider the usual matrix multiplication $\cdot$, i.e:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

(a) Show that $(\mathcal{M}, \cdot)$ is an abelian group.

(b) Compute the cyclic subgroup generated by $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}$. What is the order of M?

5. Determine whether the following statements are True or False. Justify your answer with a proof or a counterexample as appropriate.

   (a) The relation S on $\mathbb{R}$ given by xSy if and only if $x - y \in \mathbb{R} - \mathbb{N}$ is an equivalence relation.

   (b) The groups $(\mathbb{R}, +)$ and $((0, \infty), \cdot)$ are isomorphic.