Introduction to Abstract Mathematics
MAT 108 Lecture Notes

Textbook: A Transition to Advanced Mathematics, 8th Ed.
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Course Site: `quicetor.impa.br/Teaching`

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Open Sentences

A sentence that contains variables x is called an *open sentence* [we denote $P(x)$], and becomes a proposition only when its variables are assigned specific values.

The collection of objects that may be substituted to make $P(x)$ a true proposition is called the *truth set* of $P(x)$.
The truth set will always be a subset os a pre-specified ground set that we call *the universe*.
We will often use the number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ as our universes.

Example

What is the truth set of the open sentence '$x^2 < 5$' for the universes $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$, respectively?

Equivalence of open sentences

Definition

With a universe specified, two open sentences $P(x)$ and $Q(x)$ are *equivalent* iff they have the same truth set.

Example

The sentences "$3x + 2 = 20$" and "$x = 6$" are equivalent open sentences in any of the number systems we have named.

On the other hand, "$x^2 = 4$" and "$x = 2$" are *not* equivalent when the universe is $\mathbb{R}$. They *are* equivalent when the universe is $\mathbb{N}$.

Existential Quantifier

Definition

The symbol $\exists$ is called the *existential quantifier*.
For an open sentence $P(x)$, the sentence $(\exists x)P(x)$ is read "There exists x such that $P(x)$" or "For some x, $P(x)$." The sentence $(\exists x)P(x)$ is true iff the truth set of $P(x)$ is nonempty.

Example

Determine the truth values of these statements for the universe $\mathbb{R}$:

a) $(\exists x)(x \geq 3)$
b) $(\exists x)(x^2 = -1)$
c) $(\exists x)(x^2 = -1 \vee x < 0)$

Universal Quantifier

Definition

The symbol $\forall$ is called the *universal quantifier*.
For an open sentence $P(x)$, the sentence $(\forall x)P(x)$ is read "For all x, $P(x)$" and is true iff the truth set of $P(x)$ is the entire universe.

Example

For the universe of all real numbers,
$(\forall x)(x + 2 > x)$ is true.
$(\forall x)(x > 0 \vee x = 0 \vee x < 0)$ is true. That is, every real number is positive, zero or negative.
$(\forall x)(x \geq 3)$ is false because there are (many) real numbers x for which $x \geq 3$ is false.
$(\forall x)(|x| > 0)$ is false, because 0 is not in the truth set.

There are many ways to express a quantified sentence in English.

Example

The sentence "Some real numbers have a multiplicative inverse" could be symbolized
$(\exists x)(x$ is a real number $\wedge$ x has a real multiplicative inverse$)$.
However, "x has an inverse" means there is some number that is an inverse for x (hidden quantifier), so a more complete symbolic translation is
$(\exists x)[x$ is a real number $\wedge$ $(\exists y)(y$ is a real number $\wedge$ $xy = 1)]$.

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Example

One correct translation of "Some integers are even and some integers are odd" is

$$(\exists x)(x \text{ is even}) \wedge (\exists x)(x \text{ is odd})$$

Equivalence

Definition

Two quantified sentences are *equivalent in a given universe* iff they have the same truth value in that universe. Two quantified sentences are equivalent iff they are *equivalent* in every universe.

Theorem (1.3.1)

If $A(x)$ is an open sentence with variable x, then

a) $\sim (\forall x)A(x)$ *is equivalent to* $(\exists x) \sim A(x)$.

b) $\sim (\exists x)A(x)$ *is equivalent to* $(\forall x) \sim A(x)$.

Example

For the universe of all real numbers, find a denial of "Every positive real number has a multiplicative inverse."

Unique Existential Quantifier

Definition

The symbol $\exists!$ is called the *unique existential quantifier*.
For an open sentence $P(x)$, the proposition $(\exists! x)P(x)$ is read "*there exists a unique x such that $P(x)$*" and is true iff the truth set of $P(x)$ has exactly one element.

Theorem

If $A(x)$ is an open sentence with variable x, then

a) $(\exists! x)A(x) \Rightarrow (\exists x)A(x)$.

b) $(\exists! x)A(x)$ *is equivalent to*
$(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \Rightarrow y = z)$.

Proof: Exercise!

Question

Basic Proof Methods I

In mathematics, a *theorem* is a statement that describes a pattern or relationship among quantities or structures.

A *proof* is a justification of the truth of a theorem.

We cannot define all terms nor prove all statements from previous ones. Thus, we begin with an initial set of statements, called *axioms*, that are assumed to be true.

The following rules provide guidance about what statements are allowed in a proof, and when:

The 5 Rules

First Rule: At any time state an assumption, an axiom, or a previously proved result.

Tautology Rule: At any time state a sentence whose symbolic translation is a tautology.

Replacement Rule: At any time state a sentence equivalent to any statement earlier in the proof.

The 5 Rules

Definitions Rule: At any time use a definition to state an equivalent to a statement earlier in the proof.

Modus Ponens Rule: At any time after P and $P \Rightarrow Q$ appear in a proof, state that Q is true.

Using Modus Ponens

Example

You are at a crime scene and have established the following facts:

a) If the crime did not take place in the billiard room, then Colonel Mustard is guilty.

b) The lead pipe is not the weapon.

c) Either Colonel Mustard is not guilty or the weapon used was a lead pipe.

Show that the crime took place in the billiard room.

Direct Proof

DIRECT PROOF OF $P \Rightarrow Q$
Proof:
Assume P.
$\vdots$
Therefore, Q. Thus, $P \Rightarrow Q$. $\Diamond$

Example

Let x be an integer. Prove that if x is odd, then $x + 1$ is even.

Definition

Theorem (1.4.1)

Suppose a, b, and c are integers. Prove that if a divides b and b divides c, then a divides c.

Tips for writing direct proofs

When developing a direct proof of a conditional sentence, use the following strategy:

1. Determine precisely the hypotheses (if any) and the antecedent and consequent.
2. Replace (if necessary) the antecedent with a more usable equivalent.
3. Replace (if necessary) the consequent by something equivalent and more readily shown.
4. Beginning with the assumption of the antecedent, develop a chain of statements that leads to the consequent. Each statement in the chain must be deducible from its predecessors or other known results.

Working backwards

Sometimes, working backwards first is useful.

Example

Prove that if $x^2 \leq 1$, then $x^2 - 7x > -10$.

Strategies when dealing with compound propositions

We have learned how to construct proofs of statements of the form $(P \wedge Q) \Rightarrow R$.

A proof of a statement symbolized by $P \implies (Q \wedge R)$ would probably have two parts. In one part we prove $P \implies Q$ and in the other part we prove $P \implies R$.

To prove a conditional sentence of the form $P \Rightarrow (Q \vee R)$, one often proves either the equivalent $(P \wedge \sim Q) \Rightarrow R$ or the equivalent $(P \wedge \sim R) \Rightarrow Q$.

Strategies when dealing with compound propositions

A statement of the form $(P \vee Q) \Rightarrow R$ has the meaning: "If either P is true or Q is true, then R is true". A natural way to prove such a statement is by cases, so the proof outline would have the form:

- Case 1. Assume P. . . . Therefore R.
- Case 2. Assume Q. . . . Therefore R.