Introduction to Abstract Mathematics
MAT 108 Lecture Notes

Textbook: A Transition to Advanced Mathematics, 8th Ed.
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Course Site: `quicetor.impa.br/Teaching`

Canvas: Only for Zoom, Videos and Grading
Zoom Meeting: https://ucdavis.zoom.us/j/99215204274

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Contents

Working backwards

Sometimes, working backwards first is useful.

Example

Prove that if $x^2 \leq 1$, then $x^2 - 7x > -10$.

Strategies when dealing with compound propositions

We have learned how to construct proofs of statements of the form $(P \wedge Q) \Rightarrow R$.

A proof of a statement symbolized by $P \implies (Q \wedge R)$ would probably have two parts. In one part we prove $P \implies Q$ and in the other part we prove $P \implies R$.

To prove a conditional sentence of the form $P \Rightarrow (Q \vee R)$, one often proves either the equivalent $(P \wedge \sim Q) \Rightarrow R$ or the equivalent $(P \wedge \sim R) \Rightarrow Q$.

Strategies when dealing with compound propositions

A statement of the form $(P \vee Q) \Rightarrow R$ has the meaning: "If either P is true or Q is true, then R is true". A natural way to prove such a statement is by cases, so the proof outline would have the form:

- Case 1. Assume P. . . . . Therefore R.
- Case 2. Assume Q. . . . . Therefore R.

Proof by contraposition

PROOF BY CONTRAPOSITION OF $P \Rightarrow Q$

Proof.

Assume $\sim Q$.

$\vdots$

Therefore, $\sim P$.

Thus, $\sim Q \Rightarrow \sim P$

Therefore, $P \Rightarrow Q$. $\Diamond$

Proof by contradiction

PROOF OF P BY CONTRADICTION

Proof.

Suppose $\sim P$.

$\vdots$

Therefore, Q.

$\vdots$

Therefore, $\sim Q$.

Hence, $Q \wedge \sim Q$ a contradiction.

Thus, P. $\square$

Example

$\sqrt{2}$ *is an irrational number.*

Proofs of biconditional sentences

TWO-PART PROOF OF $P \iff Q$

Proof.

i) Show $P \Rightarrow Q$.

ii) Show $Q \Rightarrow P$.

Therefore, $P \iff Q$. $\square$

Proofs of biconditional sentences

BICONDITIONAL PROOF OF $P \iff Q$

Proof.

P iff R_1

iff R_2

$\dots$

iff R_n

iff Q. $\square$

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Proofs Involving Quantifiers

DIRECT PROOF OF $(\forall x)P(x)$

Proof.

Let x be an arbitrary object in the universe. (The universe should be named or its objects described.)

$\vdots$

Hence $P(x)$ is true.

Since x is arbitrary, $(\forall x)P(x)$ is true. $\square$

Proof by contradiction

PROOF OF $(\forall x)P(x)$ BY CONTRADICTION

Proof.

Suppose $\sim (\forall x)P(x)$.

Then $(\exists x) \sim P(x)$.

Let t be an object such that $\sim P(t)$.

$\vdots$

Therefore $Q \wedge \sim Q$.

Thus $(\exists x) \sim P(x)$ is false, so $(\forall x)P(x)$ is true. $\square$

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Proof by contradiction

PROOF OF $(\exists x)P(x)$ BY CONTRADICTION
Proof.
Suppose $\sim (\exists x)P(x)$.
Then $(\forall x) \sim P(x)$.
$\vdots$
Therefore, $\sim Q \wedge Q$, a contradiction.
Thus $\sim (\exists x)P(x)$ is false.
Therefore $(\exists x)P(x)$ is true.$\square$

Unique existence

PROOF OF $(\exists! x)P(x)$
Proof.

i) Prove that $(\exists x)P(x)$ is true. Use any method.
ii) Prove that $(\forall y)(\forall z)[P(y) \wedge P(z) \Rightarrow y = z]$.

Assume that y and z are objects in the universe such that $P(y)$ and $P(z)$ are true.

$\vdots$

Therefore, $y = z$.
From i) and ii) conclude that $(\exists! x)P(x)$ is true. $\square$

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