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Product of Sets

- The *ordered pair* (a, b) is an object formed from two objects a and b where a is called the first coordinate and b the second coordinate. Two ordered pairs (a, b) and (c, d) are *equal* whenever their corresponding coordinates are equal; that is, when $a = c$ and $b = d$.
  Thus,$(3, 7) \neq (7, 3)$ even though the sets $\{3, 7\}$ and $\{7, 3\}$ are equal.
- We also say the *ordered n-tuples* $(a_1, a_2, \ldots, a_n)$ and $(c_1, c_2, \ldots, c_n)$ are equal iff $a_i = c_i$ for $i = 1, 2, \ldots, n$.
  Thus the ordered 5-tuples $(4, 9, 5, 0, 1), (5, 4, 9, 0, 1)$ and $(0, 1, 4, 5, 9)$ are all different.

Definition

Let A and B be sets. The *product* (or *cross product*) of A and B is

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Example

If $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, then
$A \times B = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$.

Thus $(1, 2) \in A \times B, (2, 1) \notin A \times B$, and $\{(1, 3), (2, 2)\} \subseteq A \times B$.

In this example, is $A \times B = B \times A$?
$B \times A =$

- The product of three or more sets is defined similarly. For example, for sets A, B, and C,
  $A \times B \times C = \{(a, b, c) : a \in A, b \in B \text{ and } c \in C\}$.

Properties of the product

Theorem (2.2.3)

If A, B, C, and D are sets, then

a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

c) $A \times \varnothing = \varnothing$

d) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

e) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

f) $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

Indexed Families of Sets

A set of sets is often called a *family* or a *collection* of sets.

Our next goal is to extend the definitions of union and intersection to families of sets. We will use script letters, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, ... to denote families of sets. For example,

$$\mathscr{A} = \{\{1,2,3\},\{3,4,5\},\{3,6\},\{2,3,6,7,9,10\}\}$$

is a family consisting of four sets.

The set $\mathscr{C} = \{(-x, x) : x \in \mathbb{R} \text{ and } x > 0\}$ is an infinite family of open intervals. The sets $(-1,1), (-2,2)$, and $(5,5)$ belong to $\mathscr{C}$.

Question: Does $\varnothing \in \mathscr{C}$?

Union and Intersection

Definition

The *union of the family* $\mathscr{A}$ (or the *union over* $\mathscr{A}$) is

$$\bigcup_{A\in\mathscr{A}} A = \{x : x \in A \text{ for some } A \in \mathscr{A}\}.$$

Definition

The *intersection of the family* $\mathscr{A}$ (or the intersection over $\mathscr{A}$) is

$$\bigcap_{A\in\mathscr{A}} A = \{x : x \in A \text{ for every } A \in \mathscr{A}\}.$$

Basic properties

Theorem (2.3.1)

Let $\mathscr{A}$ be a family of sets.

(a) *For every set $B \in \mathscr{A}$,* $\bigcap_{A\in\mathscr{A}} A \subseteq B$.

(b) *For every set $B \in \mathscr{A}$,* $B \subseteq \bigcup_{A\in\mathscr{A}} A$.

(c) *If $\mathscr{A} \neq \varnothing$, then* $\bigcap_{A\in\mathscr{A}} A \subseteq \bigcup_{A\in\mathscr{A}} A$.

(d) $\left(\bigcap_{A\in\mathscr{A}} A\right)^c = \bigcup_{A\in\mathscr{A}} A^c$.

(e) $\left(\bigcup_{A\in\mathscr{A}} A\right)^c = \bigcap_{A\in\mathscr{A}} A^c$.

Theorem (2.3.2)

Let $\mathscr{A} \neq \varnothing$ be a family of sets and B be a set.

(a) *If $B \subseteq A$ for all $A \in \mathscr{A}$, then $B \subseteq \bigcap_{A \in \mathscr{A}} A$.*

(b) *If $A \subseteq B$ for all $A \in \mathscr{A}$, then $\bigcup_{A \in \mathscr{A}} A \subseteq B$.*

Example

Let $\mathscr{B} = \{[-r, r^2 + 1) : r \in \mathbb{R} \text{ and } r \geq 0\}$. Calculate $\bigcap_{B \in \mathscr{B}} B$ and $\bigcup_{B \in \mathscr{B}} B$.

One identifying tag (or index) for each set

Definition

Let $\Delta \neq \varnothing$ be a set such that for each $\alpha \in \Delta$ there is a corresponding set A_α.
The family $\{A_\alpha : \alpha \in \Delta\}$ is an *indexed family of sets*.
Δ is called the *indexing set*, and each $\alpha \in \Delta$ is an *index*.

Indexed notation

For an indexed family $\mathscr{A} = \{A_\alpha : \alpha \in \Delta\}$, we write

$$\bigcup_{\alpha\in\Delta} A_\alpha = \bigcup_{A\in\mathscr{A}} A \quad \text{and} \quad \bigcap_{\alpha\in\Delta} A_\alpha = \bigcap_{A\in\mathscr{A}} A.$$

Thus,

$$x \in \bigcup_{\alpha\in\Delta} A_\alpha \iff (\exists \alpha \in \Delta)(x \in A_\alpha),$$

$$x \in \bigcap_{\alpha\in\Delta} A_\alpha \iff (\forall \alpha \in \Delta)(x \in A_\alpha).$$

Example

For each $r \in \mathbb{R}$, consider the interval (set) $B_r = [r^2, r^2 + 1]$. Calculate (i) $\bigcap_{r\in\mathbb{R}} B_r$, and (ii) $\bigcup_{r\in\mathbb{R}} B_r$.

Mathematical Induction

Principle of Mathematical Induction (PMI)

Let S be a subset of $\mathbb{N}$ with these two properties:

(i) $1 \in S$; and

(ii) $\forall n \in \mathbb{N}$, if $n \in S$, then $n+1 \in S$. (S is an *inductive set*)

Then $S = \mathbb{N}$.

Inductive definitions follow the form of PMI. For example, the sigma notation:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

is inductively defined by

- $\sum_{i=1}^{1} x_i = x_1$.
- $\forall n \in \mathbb{N},\ \sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n} x_i + x_{n+1}$.

And the product notation:

$$\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

is inductively defined by

- $\prod_{i=1}^{1} x_i = x_1$.
- $\forall n \in \mathbb{N},\ \prod_{i=1}^{n+1} x_i = \left(\prod_{i=1}^{n} x_i\right) \cdot x_{n+1}$.

Example

The noninductive definition of the *factorial* of $n \in \mathbb{N}$ is

$$n! = n(n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = \prod_{k=1}^{n} k.$$

So that $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. And its inductive definition is:

Example (Applying PMI)

For every $n \in \mathbb{N}$, guess a simplification of the following expression (i.e. compute the sum). Then show that your guess is correct:

$$1 + 3 + 5 + \cdots + (2n - 1) =$$

PROOF OF $(\forall n \in \mathbb{N})P(n)$ BY MATHEMATICAL INDUCTION

Proof: **(i) Basis Step**.

Verify that $P(1)$ is T.

(ii) Inductive Step.

Suppose $P(n)$ is T (assume the hypothesis of induction).

$\vdots$

Therefore, $P(n+1)$ is T.

Thus, by the PMI, $(\forall n \in \mathbb{N})P(n)$ is T. $\Diamond$

Example

Prove that $n + 3 < 5n^2$, $\forall n \in \mathbb{N}$.

- Show that $\forall n \in \mathbb{N}$, the polynomial $x - y$ divides $x^n - y^n$.

Example

Prove that $\forall n \in \mathbb{N}$, $\prod_{k=1}^{n}(4k-2) = \dfrac{(2n)!}{n!}$.

Given a fulcrum and a long enough lever, Archimedes could move the world

Theorem (Archimedean Principle for $\mathbb{N}$)

For all $a, b \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that $sb > a$.