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Given a fulcrum and a long enough lever, Archimedes could move the world

Theorem (Archimedean Principle for $\mathbb{N}$)

For all $a, b \in \mathbb{N}$, *there exists* $s \in \mathbb{N}$ *such that* $sb > a$.

Generalized PMI

Generalized Principle of Mathematical Induction (GPMI)

Useful for statements of the form $(\exists k \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq k \Rightarrow P(n))$

Proof: **(i) Basis Step**.

Verify that $P(k)$ is T.

(ii) Inductive Step.

Suppose $n \geq k$ and $P(n)$ is T (hypothesis of induction).

$\vdots$

Therefore, $P(n+1)$ is T.

Thus, by the GPMI, $P(n)$ is T, $\forall n \in \mathbb{N}$ such that $n \geq k$. $\Diamond$

Example

Prove by induction that $n^2 - n - 20 > 0$ for all $n > 5$.

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Equivalent Forms of Induction

PRINCIPLE OF COMPLETE (or Strong) INDUCTION (PCI)

We want to prove that $(\exists k \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq k \Rightarrow P(n))$

Proof: **(i) Basis Step**.

Verify that $P(k)$ is T.

(ii) Inductive Step.

Suppose $n > k$ and $P(k), P(k+1), \ldots, P(n-1)$ are all T.

$\vdots$

Therefore, $P(n)$ is T.

Thus, by the PCI, $P(n)$ is T, $\forall n \in \mathbb{N}$ such that $n \geq k$. $\Diamond$

Example

Every $m \in \mathbb{N}$ greater than 1 is prime or is a product of primes.

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The WOP

WELL-ORDERING PRINCIPLE (WOP)

Every nonempty subset of $\mathbb{N}$ has a smallest element.

Example

Every $m \in \mathbb{N}$ greater than 1 is prime or is a product of primes.

The Division Algorithm

Theorem (2.5.2)

For all $a, b \in \mathbb{Z}$ *with* $a \neq 0$, $\exists! q \in \mathbb{Z}$ *and* $\exists! r \in \mathbb{Z}$ *such that*

$$b = aq + r \quad \text{and} \quad 0 \leq r < |a|.$$

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Relations and Partitions

Definition

Let A and B be sets. R is a *relation from A to B* iff $R \subseteq A \times B$.
A relation from A to A is called a *relation on A*.
If $(a, b) \in R$, we write aRb and say a is *R-related* (or simply *related*) to b. If $(a, b) \notin R$, we write $a \not R b$.

Example

Let $A = \{-1, 2, 3, 4\}$ and $B = \{1, 2, 4, 5, 6\}$. The set of ordered pairs

$$R = \{(-1, 5), (2, 4), (2, 1), (4, 2)\}$$

is a relation from A to B. We can also describe R by writing $-1R5, 2R4, 2R1$ and $4R2$.

Definition

For any set A, the *identity relation on A* is the set

$$I_A = \{(a, a) : a \in A\}.$$

Definition

The *domain* of the relation R from A to B is the set
$Dom(R) = \{x \in A\colon \text{ there exists } y \in B \text{ such that } xRy\}$.
The *range* of the relation R is the set
$Rng(R) = \{y \in B\colon \text{ there exists } x \in A \text{ such that } xRy\}$.

Example

Let $T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + 4y^2 \leq 16\}$. Prove that

i) $Dom(T) = [-4, 4]$ and

ii) $Rng(T) = [-2, 2]$.

Inverse relation

Definition

If R is a relation fron A to B, then the *inverse* of R is the relation

$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$

- Let $T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < 4x^2 - 7\}$. Find T^{-1}.

Theorem (3.1.1)

Let R be a relation from A to B.

a) $Dom(R^{-1}) = Rng(R)$.
b) $Rng(R^{-1}) = Dom(R)$.

Composite relations

Definition

Let R be a relation from A to B, and let S be a relation from B to C. The *composite* of R and S is
$S \circ R = \{(a, c)\colon \exists b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$.

Example

Let $A = B = C = \mathbb{N}$, $R = \{(1, 28)\}$, $S = \{(28, 1)\}$. Is $S \circ R = R \circ S$?

Example

Consider the sets $A = \{1, 2, 3, 4, 5\}$, $B = \{p, q, r, s, t\}$ and $C = \{x, y, z, w\}$, and the relations

$$R = \{(1, p), (1, q), (2, q), (3, r), (4, s)\},$$

$$S = \{(p, x), (q, x), (q, y), (s, z), (t, z)\}.$$

Calculate $S \circ R, (S \circ R) \circ R^{-1}$ and $(R^{-1} \circ S^{-1}) \circ (S \circ R)$.

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