CONTENTS

July 7, 2020

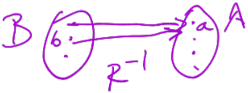
$R \subseteq A \times B$

Definition

If R is a relation fron A to B, then the *inverse* of R is the relation

$$R^{-1} = \{(y,x) : (x,y) \in R\}. \subseteq B \times A$$

$R \subseteq A \times B$



Theorem (3.1.1)

Let R be a relation from A to B.

a) $Dom(R^{-1}) = Rng(R)$.

b) $Rng(R^{-1}) = Dom(R)$.

Proof (a) $b \in Dom(R^{-1}) \iff \exists a \in A$ s.t. $(b,a) \in R^{-1}$

$\iff (a,b) \in R$ for some $a \in A$

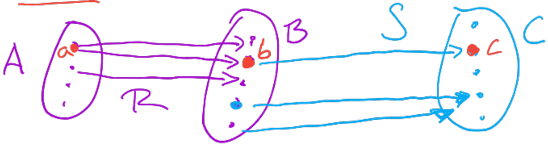
$\iff b \in Rng(R)$. #

(b) Exercise!

Composite relations

Definition

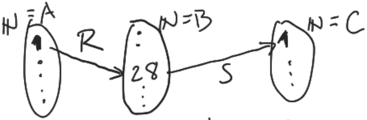
Let R be a relation from A to B, and let S be a relation from B to C. The *composite* of R and S is
$S \circ R = \{(a, c)\colon \exists b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$.



$R \subseteq A \times B, \quad S \subseteq B \times C$

Example

Let $A = B = C = \mathbb{N}$, $R = \{(1, 28)\}$, $S = \{(28, 1)\}$. Is $S \circ R = R \circ S$?



$\neq$

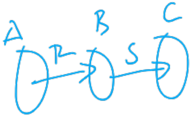
$S \circ R = \{(1,1)\} = I_D$

where $D = \{1\}$

$R \circ S = I_{\{28\}}$.

Example

Consider the sets $A = \{1, 2, 3, 4, 5\}$, $B = \{p, q, r, s, t\}$ and $C = \{x, y, z, w\}$, and the relations

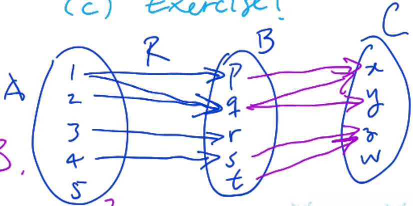


$$R = \{(1, p), (1, q), (2, q), (3, r), (4, s)\},$$

$$S = \{(p, x), (q, x), (q, y), (s, z), (t, z)\}.$$

Find $\underbrace{S \circ R}_{(a)}$, $\underbrace{(S \circ R) \circ R^{-1}}_{(b)}$ and $\underbrace{(R^{-1} \circ S^{-1}) \circ (S \circ R)}_{(c)}$. Exercise!

Sol (a) We look for all possible ways to go from A to C, by using arrows connected at B.

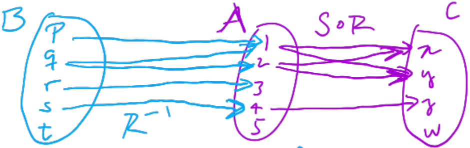


$S \circ R = \{(1, x), (1, y), (2, x), (2, y), (4, z)\}$

$\subseteq A \times C$

(b) $R \subseteq A \times B$

$R^{-1} \subseteq B \times A$



So, $(S \circ R) \circ R^{-1}$ is a relation from B to C, given by

$$(S \circ R) \circ R^{-1} = \{(p,x), (p,y), (q,x), (q,y), (s,z)\}$$

(c) $(R^{-1} \circ S^{-1}) \circ (S \circ R)$ . . . -

Theorem (3.1.2)

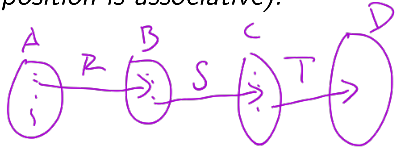
Suppose A, B, C, and D are sets. Let R be a relation from A to B, S be a relation from B to C, and T be a relation from C to D.

a) $(R^{-1})^{-1} = R$.

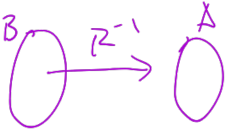
b) $T \circ (S \circ R) = (T \circ S) \circ R$, *(composition is associative).*

c) $I_B \circ R = R$ *and* $R \circ I_A = R$.

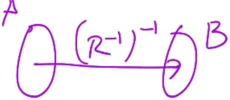
d) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.



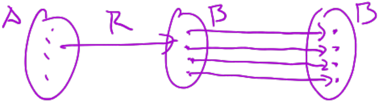
(a)



so,

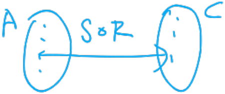


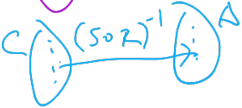
(c)



$I_B \circ R \subseteq A \times B$.

(d)





Exercise!

Equivalence relations

§3.2

Definition

Let A be a set and R be a relation on A. $R \subseteq A \times A$

① R is *reflexive on* A iff for all $x \in A$, $x \; R \; x$. $\Leftrightarrow (x,x) \in R, \; \forall x \in A$.

② R is *symmetric* iff for all $x, y \in A$, if $x \; R \; y$, then $y \; R \; x$. $(x,y) \in R \Rightarrow (y,x) \in R$

③ R is *transitive* iff for all $x, y, z \in A$, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$.
$(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$.

Example

Consider the relation '$>$' on $A = \mathbb{N}$. $x, y \in \mathbb{N}$, $x R y$ iff $x > y$

① Not reflexive, since $1 \in \mathbb{N}$ but $1 \not> 1$ so $(1,1) \notin R$.

② Not symmetric, since $x = 2$, $y = 1$, $x > y$ $(x,y) \in R$ but $(y,x) \notin R$.
$(2,1) \in R$ but $(1,2) \notin R$

③ It is transitive! Since $x > y$ and $y > z \Rightarrow x > z$, $\forall x, y, z \in \mathbb{N}$.

Definition

A relation R on a set A is an *equivalence relation* on A if R is reflexive on A, symmetric, and transitive.

Example

Prove that $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^3 = y^3\}$ is an equivalence relation on $\mathbb{R}$.

Theorem (3.2.1)

Let A be a set. For $\mathscr{P}(A)$, the relation "is a subset of" is reflexive on $\mathscr{P}(A)$ and transitive but not symmetric.

UCDAVIS
AGGIES