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Example

Prove that $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^2\}$ is an equivalence relation on $\mathbb{R}$.

Theorem (3.2.1)

Let A be a set. For $\mathscr{P}(A)$, the relation 'is a subset of' is reflexive on $\mathscr{P}(A)$ and transitive but not symmetric.

Example

Prove that the relation, $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + y \text{ is even}\}$ is an equivalence relation on $\mathbb{Z}$.

Equivalence classes

Definition

Let R be an equivalence relation on a set A. For $x \in A$, the *equivalence class of* x *modulo* R is the set

$$\bar{x} = \{y \in A : x \; R \; y\}.$$

Each element of $\bar{x}$ is called a *representative* of this class. The set

$$A/R = \{\bar{x} : x \in A\}$$

of all equivalence classes is called A *modulo* R

Example

The relation $H = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$ is an equivalence relation on the set $A = \{1,2,3\}$. Here

Thus, $A/H = \{\bar{1},\bar{3}\} = \{\{1,2\},\{3\}\}$.

Example

We know that $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^2\}$ is a equivalence relation on $\mathbb{R}$. Describe $\mathbb{R}/S$.

Theorem (3.2.2)

Let R be an equivalence relation on a set $A \neq \varnothing$. For all $x, y \in A$,

a) $x \in \bar{x}$ *and* $\bar{x} \subseteq A$.

b) xRy *if and only if* $\bar{x} = \bar{y}$.

c) $x \not R y$ *if and only if* $\bar{x} \cap \bar{y} = \varnothing$.

Congruence mod m on $\mathbb{Z}$

Definition

Fix $m \in \mathbb{N}$. For $x, y \in \mathbb{Z}$, we say x *is congruent to* y *modulo* m iff $m|(x-y)$. We write $x = y(mod\ m)$.
The number m is called the *modulus* of the congruence.

Theorem (3.2.3)

For every fixed $m \in \mathbb{N}$, congruence modulo m is an equivalence relation on $\mathbb{Z}$.

Theorem (3.2.4)

Let m be a fixed positive integer. Then

1. *For $x, y \in \mathbb{Z}$, $x = y \pmod{m}$ iff the remainder when x is divided by m equals the remainder when y is divided by m.*
2. *$\mathbb{Z}_m$ consists of m distinct equivalence classes:* $\mathbb{Z}_m = \{\bar{0}, \bar{1}, \bar{2}, \cdots, \overline{m-1}\}$.

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Partitions

Definition

Let A be a nonempty set. $\mathscr{P}$ is a *partition of A* iff $\mathscr{P}$ is a set of subsets of A such that

i) If $X \in \mathscr{P}$, then $X \neq \varnothing$.

ii) If $X \in \mathscr{P}$ and $Y \in \mathscr{P}$, then $X = Y$ or $X \cap Y = \varnothing$.

iii) $\bigcup_{X \in \mathscr{P}} X = A$.

Example

Prove that the family $\mathscr{G} = \{[n, n+1) : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$.

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