Chapter 5 July 20, 2020

Cardinality

EQUIVALENT SETS

Definition

Two sets A and B are *equivalent* if $\exists\, f : A \xrightarrow[onto]{1-1} B$.

The sets are also said to be *in one-to-one corresponce*, and we write $A \approx B$.

If A and B are not equivalent, we write $A \not\approx B$.

Theorem (5.1.1)

$\approx$ *is an equivalence relation on the class of all sets.*

Lemma (5.1.2)

Let A, B, C and D be sets with $A \approx C$ and $B \approx D$.

1. *If $A \cap B = \varnothing$ and $C \cap D = \varnothing$, then $A \cup B \approx C \cup D$.*
2. *$A \times B \approx C \times D$.*

Finite sets

Definition

For $k \in \mathbb{N}$, denote $\mathbb{N}_k = \{1, 2, \ldots, k\}$.
A set S is *finite* if $S = \varnothing$ or $S \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$.
A set S is *infinite* if it is not finite.

Definition

Let S be a finite set.
If $S = \varnothing$, then S has *cardinality* 0. We write $\bar{\bar{S}} = 0$.
If $\exists k \in \mathbb{N}$ s.t. $S \approx \mathbb{N}_k$, then S has *cardinality* k. We write $\bar{\bar{S}} = k$.

Theorem (5.1.3)

If A is finite and $B \approx A$, then B is finite.

Lemma (5.1.4)

If S is a finite set with cardinality k and $x \notin S$, then $S \cup \{x\}$ is finite and has cardinality $k+1$.

Lemma (5.1.5)

$\forall k \in \mathbb{N}$, *every subset of* $\mathbb{N}_k$ *is finite.*

Theorem (5.1.6)

Every subset of a finite set is finite.

Theorem (5.1.7)

Let $A_1, A_2, \ldots, A_n, A$ and B be finite sets.

1. *If $A \cap B = \varnothing$, then $A \cup B$ is finite and $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}}$.*
2. *More generally, $A \cup B$ is finite and $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}}$.*
3. *$\bigcup_{i=1}^{n} A_i$ is finite.*

Lemma (5.1.8)

Fix $r \in \mathbb{N}$ *with* $r \geq 2$. $\forall x \in \mathbb{N}_r$, $\mathbb{N}_r - \{x\} \approx \mathbb{N}_{r-1}$.

The Pigeonhole Principle

Theorem (5.1.9)

Let $n, r \in \mathbb{N}$ *and* $f : \mathbb{N}_n \to \mathbb{N}_r$. *If* $n > r$, *then* f *is not 1-1.*

Corollary (5.1.10)

If $n > r, \bar{\bar{A}} = n$ and $\bar{\bar{B}} = r$, then there is no 1-1 function $f : A \to B$.

Corollary (5.1.11)

If A is finite, then A is not equivalent to any of its proper subsets.

INFINITE SETS

Theorem (5.2.1)

The set $\mathbb{N}$ of natural numbers is infinite.

Denumerable sets

Definition

The set S is *denumerable* if $S \approx \mathbb{N}$. For a denumerable set S, we say S has *cardinal number* $\aleph_0$ (or *cardinality* $\aleph_0$) and write $\bar{\bar{S}} = \aleph_0$.

Theorem (5.2.2)

The set $\mathbb{Z}$ is denumerable.

Theorem (5.2.3)

a) *The set* $\mathbb{N} \times \mathbb{N}$ *is denumerable.*

b) *If* A *and* B *are denumerable sets, then* $A \times B$ *is denumerable.*

Countable sets

Definition

A set is *countable* if S is finite or denumerable. We say S is *uncountable* if S is not countable.

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Theorem (5.2.4)

The open interval $(0, 1)$ *is uncountable.*

Definition

A set S has *cardinality* $\mathfrak{c}$ (or *cardinal number* $\mathfrak{c}$) if S is equivalent to $(0,1)$. We write $\bar{\bar{S}} = \mathfrak{c}$, which stands for *continuum*.

Theorem (5.2.5)

a) *Every open interval* (a, b) *is uncountable and has cardinality* $\mathfrak{c}$.

b) *The set* $\mathbb{R}$ *of all real numbers is uncountable and has cardinality* $\mathfrak{c}$.

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