Cardinality

Concepts of Algebra

The Ordering of Cardinal Numbers

Definition

Let A and B be sets. Then
$\bar{\bar{A}} = \bar{\bar{B}}$ iff $A \approx B$; otherwise $\bar{\bar{A}} \neq \bar{\bar{B}}$.
$\bar{\bar{A}} \leq \bar{\bar{B}}$ iff there exists a 1-1 function $f : A \to B$.
$\bar{\bar{A}} < \bar{\bar{B}}$ if $\bar{\bar{A}} \leq \bar{\bar{B}}$ and $\bar{\bar{A}} \neq \bar{\bar{B}}$.

Theorem (5.4.1)

For sets A, B, and C,

a) $\bar{\bar{A}} \leq \bar{\bar{A}}$.

b) If $\bar{\bar{A}} = \bar{\bar{B}}$ and $\bar{\bar{B}} = \bar{\bar{C}}$, then $\bar{\bar{A}} = \bar{\bar{C}}$.

c) If $\bar{\bar{A}} \leq \bar{\bar{B}}$ and $\bar{\bar{B}} \leq \bar{\bar{C}}$, then $\bar{\bar{A}} \leq \bar{\bar{C}}$.

d) $\bar{\bar{A}} \leq \bar{\bar{B}}$ iff $\bar{\bar{A}} < \bar{\bar{B}}$ or $\bar{\bar{A}} = \bar{\bar{B}}$.

e) If $A \subseteq B$, then $\bar{\bar{A}} \leq \bar{\bar{B}}$.

f) $\bar{\bar{A}} \leq \bar{\bar{B}}$ iff $\exists W \subseteq B$ such that $\overline{\overline{W}} = \bar{\bar{A}}$.

Theorem (5.4.2)

1. *For $m, n \in \mathbb{N} \cup \{0\}$, if $m < n$ as integers, then $m < n$ as finite cardinals.*
2. *Every finite cardinal is less than $\aleph_0$.*
3. *The cardinal $\aleph_0$ is less than the cardinal $\mathfrak{c}$.*

Theorem (5.4.3 Cantor's Theorem)

For every set A, $\bar{\bar{A}} < \overline{\overline{\mathscr{P}(A)}}$.

Theorem (5.4.4: Cantor–Schröder–Bernstein Theorem)

If $\bar{\bar{A}} \leq \bar{\bar{B}}$ *and* $\bar{\bar{B}} \leq \bar{\bar{A}}$*, then* $\bar{\bar{A}} = \bar{\bar{B}}$*.*

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Theorem (5.4.5)

$\overline{\overline{\mathscr{P}(\mathbb{N})}} = c.$

Theorem (5.4.6)

For sets A,B, and C,

a) *if* $\bar{\bar{A}} \leq \bar{\bar{B}}$, *then* $\bar{\bar{B}} \not< \bar{\bar{A}}$.

b) *if* $\bar{\bar{A}} \leq \bar{\bar{B}}$ *and* $\bar{\bar{B}} < \bar{\bar{C}}$, *then* $\bar{\bar{A}} < \bar{\bar{C}}$.

c) *if* $\bar{\bar{A}} < \bar{\bar{B}}$ *and* $\bar{\bar{B}} \leq \bar{\bar{C}}$, *then* $\bar{\bar{A}} < \bar{\bar{C}}$.

d) *if* $\bar{\bar{A}} < \bar{\bar{B}}$ *and* $\bar{\bar{B}} < \bar{\bar{C}}$, *then* $\bar{\bar{A}} < \bar{\bar{C}}$.

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Concepts of Algebra

Algebraic Structures

Definition

Let A be a nonempty set. A *binary operation* $*$ *on* A is a function from $A \times A$ to A. The notation for the image of $(x, y) \in A \times A$ is $x * y$ under the operation $*$.

Definition

An *algebraic system* or *algebraic structure* is a nonempty set A with a collection of one or more operations on A and a (possibly empty) collection of relations on A.

Definition

Let $(A, *)$ be an algebraic system. Let B be a subset of A. We say B is *closed under the operation* $*$ if $x * y \in B$ for all $x, y \in B$.

Definition

Let $(A, *)$ be an algebraic system. Then
$*$ is *commutative* on A if for all $x, y \in A, x * y = y * x$.
$*$ is *associative* on A if for all $x, y, z \in A, (x * y) * z = x * (y * z)$.
an element e of A is an *identity element* for $*$ if for all $x \in A$,
$x * e = e * x = x$.
if A has an identity element e and if a and b are in A, then b is an *inverse* of a if $a = b * a = e$. In this case a would also be an inverse of b.

Theorem (6.1.1)

Let $(A, *)$ *be an algebraic structure.*

Theorem (6.1.2)

For every natural number m,

a) $(\mathbb{Z}_m, +_m)$ *is an algebraic system that is associative and commutative with identity element* 0. *Every element has an inverse.*

b) $(\mathbb{Z}_m, \cdot_m)$ *is an algebraic system that is associative and commutative. If* $m > 1$, *the system has identity element* 1 .

Definition

Let a be a nonzero element of $(\mathbb{Z}_m, \cdot)$.
If $ab = 0$ for some $b \neq 0$, then we say a (and also b) is a *divisor of zero*.
If a has a multiplicative inverse in $\mathbb{Z}_m$, then a is called a *unit* in $\mathbb{Z}_m$.
The set of all units in $\mathbb{Z}_m$ is denoted ${}_m$.