1.4.9. (c) If $\underbrace{ab>0 \text{ and } bc<0}_{P} \Rightarrow \underbrace{ax^2+bx+c=0 \text{ has 2 real sols.}}_{Q}$

Sketch: $a,b,c \in \mathbb{Z}$, $x \in \mathbb{R}$. We know that the roots are

$$x_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

If $b^2-4ac=0 \Rightarrow x_+ = x_-$. we do not want it.

If $b^2-4ac<0 \Rightarrow x_+, x_- \in \mathbb{C}-\mathbb{R}$. we do not want it neither.

• If $\boxed{b^2-4ac>0} \Rightarrow x_+, x_- \in \mathbb{R}$ and $x_+ \neq x_-$. (2 sols).

$P \Rightarrow \cdots \Rightarrow ?$ $? \Rightarrow \cdots \Rightarrow Q$

P: $ab>0$ and $bc<0$

$\Rightarrow (ab)(bc)<0 \Rightarrow \boxed{b^2ac<0}$ Δ

Is it possible $b=0$? No, if $b=0 \Rightarrow b^2 \cdot ac = 0$.

So $b^2>0$, by Δ $\frac{b^2ac}{b^2} < \frac{0}{b^2} \Leftrightarrow ac<0 \Leftrightarrow 4ac<0$

$\Leftrightarrow \boxed{-4ac>0}$

Proof: let $a,b,c \in \mathbb{Z}$ and $x \in \mathbb{R}$ such that

$ab>0$ and $bc<0 \Rightarrow (ab)(bc)<0 \Rightarrow b^2ac<0$

Note that $b=0 \Rightarrow b^2ac=0$. Therefore $b \neq 0$, so

$b^2>0 \Rightarrow \frac{b^2ac}{b^2} < \frac{0}{b^2}$ that means $ac<0 \Rightarrow -4ac>0$.

since $-4<0$

Now, $b^2>0$ and $-4ac>0$ implies

$b^2+(-4ac)>0$, thus $b^2-4ac>0$.

This means that the discriminant of the eq

$ax^2+bx+c=0$ is positive so the solutions

$$x_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ are real and $x_+ \neq x_-$

Therefore, the equation has 2 real roots. #

1.5.3 (f) If $\underbrace{xy \text{ is odd}}_{P(x,y)} \Rightarrow \underbrace{\text{both } x \text{ and } y \text{ are odd}}_{Q(x,y)}$

Have to do is to show $\sim Q \Rightarrow \sim P$.

Q: "x is odd and y is odd"

$\sim Q$: "$\underbrace{x \text{ is even}}_{R}$ or $\underbrace{y \text{ is even}}_{S}$"

$(R \vee S) \Rightarrow \sim P$

Case 1: $R \Rightarrow \sim P$

Case 2: $S \Rightarrow \sim P$.

. . .

DO IT!

1.6.7i $\underbrace{(\forall \varepsilon > 0)}(\exists K \in \mathbb{N})\underbrace{(\forall x \in \mathbb{R})(x > K \Rightarrow \frac{1}{4x} < \varepsilon)}$

Proof: Fix $\varepsilon > 0$. Then we can take $K \in \mathbb{N}$

$K > \frac{1}{4\varepsilon} \Rightarrow \frac{1}{4K} < \varepsilon$.

Now fix $x \in \mathbb{R}$ s.t $x > K \Rightarrow$ . . . finish it!

#

want $\frac{1}{4K} < \varepsilon \iff \frac{1}{4\varepsilon} < K$

Sketch $x > K \Rightarrow 4x > 4K \Rightarrow \frac{1}{4x} < \frac{1}{4K}$

1.6.5(a) $x \in \mathbb{N}$. x prime $\iff \left[x > 1 \wedge \sim \underbrace{\left(\exists k \in \{2,3,\ldots\} \; k \le \sqrt{x} \wedge k \mid x \right)}_{(\forall k \in \{2,3,\ldots\})(k > \sqrt{x} \vee k \nmid x)} \right]$

Proof: $(\Rightarrow)$ Easy.

$(\Leftarrow)$ Fix $x \in \mathbb{N}$.
x prime iff $(\forall \ell \in \mathbb{N})(\ell \mid x \Rightarrow (\ell = 1 \vee \ell = x))$

Assume that $\forall k \in \{2,3,\ldots\}$ either $\underbrace{k > \sqrt{x}}_{R}$ or $\underbrace{k \nmid x}_{S}$.
Let us prove that x is prime.

DO THE FULL PROOF BY using sketch. //

$\sim(P \Rightarrow Q) \iff P \wedge \sim Q$

Sketch: (By contr), $\exists \ell \in \mathbb{N}$ $\ell \mid x \wedge \sim(\ell = 1 \vee \ell = x)$

$\ell \mid x \wedge \ell \ne 1 \wedge \ell \ne x \iff \ell \mid x \wedge \ell \ge 2 \wedge \ell \ne x$

$\exists m \in \mathbb{N}$ s.t. $x = \ell \cdot m \ge \ell$ so $\underbrace{\ell \in \{2,3,\ldots,x-1\}}$

$\boxed{\ell > \sqrt{x}}$ $\}0$ $\frac{x}{\ell} = m \ne 1$ since $x \ne \ell$ $\quad \frac{x}{\ell} \in \{2,3,\ldots\}$

either $\frac{x}{\ell} > \sqrt{x}$ or $\frac{x}{\ell} \nmid x$ F $\frac{x}{\ell} \mid x$

so $\boxed{\frac{x}{\ell} > \sqrt{x}}$ $\}0$ multipl. $\ell \cdot \frac{x}{\ell} > \sqrt{x} \cdot \sqrt{x}$

so $x > x$. $(\rightarrow\leftarrow)$