1.4.9. (c) If $\underbrace{ab>0 \text{ and } bc<0}_{P} \Rightarrow \underbrace{ax^2+bx+c=0 \text{ has 2 real sols.}}_{Q}$

Sketch: $a,b,c \in \mathbb{Z}$, $x \in \mathbb{R}$. We know that the roots are

$$x_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

If $b^2-4ac=0 \Rightarrow x_+ = x_-$. we do not want it.

If $b^2-4ac<0 \Rightarrow x_+, x_- \in \mathbb{C}-\mathbb{R}$. we do not want it neither.

• If $\boxed{b^2-4ac>0} \Rightarrow x_+, x_- \in \mathbb{R}$ and $x_+ \neq x_-$. (2 sols).

$P \Rightarrow \cdots \Rightarrow ?\quad ? \Rightarrow \cdots \Rightarrow Q$

P: $ab>0$ and $bc<0$

$\Rightarrow (ab)(bc)<0 \Rightarrow \boxed{b^2ac<0}$ (△)

Is it possible $b=0$? No, if $b=0 \Rightarrow b^2 \cdot ac = 0$.

So $b^2>0$, by (△) $\frac{b^2ac}{b^2} < \frac{0}{b^2} \Leftrightarrow ac<0 \Leftrightarrow 4ac<0$.

$\Leftrightarrow \boxed{-4ac>0}$

Proof: let $a,b,c \in \mathbb{Z}$ and $x \in \mathbb{R}$ such that

$ab>0$ and $bc<0 \Rightarrow (ab)(bc)<0 \Rightarrow b^2ac<0$

Note that $b=0 \Rightarrow b^2ac=0$. Therefore $b \neq 0$, so

$b^2>0 \Rightarrow \frac{b^2ac}{b^2} < \frac{0}{b^2}$ that means $ac<0 \Rightarrow -4ac>0$.

since $-4<0$

Now, $b^2>0$ and $-4ac>0$ implies

$b^2+(-4ac)>0$, thus $b^2-4ac>0$.

This means that the discriminant of the eq $ax^2+bx+c=0$ is positive so the solutions

$$x_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

are real and $x_+ \neq x_-$

Therefore, the equation has 2 real roots. #

1.5.3 (f) If $\underbrace{xy \text{ is odd}}_{P(x,y)} \Rightarrow \underbrace{\text{both } x \text{ and } y \text{ are odd}}_{Q(x,y)}$

Have to do is to show $\sim Q \Rightarrow \sim P$.

Q: "x is odd and y is odd"

$\sim$Q: "$\underbrace{x \text{ is even}}_{R}$ or $\underbrace{y \text{ is even}}_{S}$"

$(R \vee S) \Rightarrow \sim P$

Case 1: $R \Rightarrow \sim P$

Case 2: $S \Rightarrow \sim P$.

. . . DO IT!

1.6.7i $\underbrace{(\forall \varepsilon > 0)}(\exists K \in \mathbb{N}) \underbrace{(\forall x \in \mathbb{R})\left(x > K \Rightarrow \frac{1}{4x} < \varepsilon\right)}$

Proof: Fix $\varepsilon > 0$. Then we can take $K \in \mathbb{N}$

$K > \frac{1}{4\varepsilon} \Rightarrow \frac{1}{4K} < \varepsilon$.

Now fix $x \in \mathbb{R}$ s.t $x > K \Rightarrow$ . . . finish it!

#

want $\frac{1}{4K} < \varepsilon \iff \frac{1}{4\varepsilon} < K$

Sketch $x > K \Rightarrow 4x > 4K \Rightarrow \frac{1}{4x} < \frac{1}{4K}$

1.6.5(a) $x \in \mathbb{N}$. x prime $\iff \left[x > 1 \wedge \sim \underbrace{\left(\exists k \in \{2,3,\dots\} \; k \le \sqrt{x} \wedge k \mid x \right)}_{(\forall k \in \{2,3,\dots\})(k > \sqrt{x} \vee k \nmid x)} \right]$

Proof: ($\Rightarrow$) Easy.

($\Leftarrow$) Fix $x \in \mathbb{N}$.

x prime iff $(\forall \ell \in \mathbb{N})(\ell \mid x \Rightarrow (\ell = 1 \vee \ell = x))$

Assume that $\forall k \in \{2,3,\dots\}$ either $\underbrace{k > \sqrt{x}}_{R}$ or $\underbrace{k \nmid x}_{S}$.

Let us prove that x is prime.

DO THE FULL PROOF BY using sketch. #

$\sim(P \Rightarrow Q) \iff P \wedge \sim Q$

Sketch: (By contr). $\exists \ell \in \mathbb{N}$ $\ell \mid x \wedge \sim(\ell = 1 \vee \ell = x)$

$\ell \mid x \wedge \ell \neq 1 \wedge \ell \neq x \iff \ell \mid x \wedge \ell \ge 2 \wedge \ell \neq x$

$\exists m \in \mathbb{N}$ s.t. $x = \ell \cdot m \ge \ell$ so $\underbrace{\ell \in \{2,3,\dots,x-1\}}$

$\boxed{\ell > \sqrt{x}} \; \}0$ $\quad \frac{x}{\ell} = m \neq 1$ since $x \neq \ell$ $\quad \frac{x}{\ell} \in \{2,3,\dots\}$

either $\frac{x}{\ell} > \sqrt{x}$ or $\frac{x}{\ell} \nmid x$ F $\frac{x}{\ell} \mid x$

so $\boxed{\frac{x}{\ell} > \sqrt{x}} \; \}0$ multipl. $\ell \cdot \frac{x}{\ell} > \sqrt{x} \cdot \sqrt{x}$

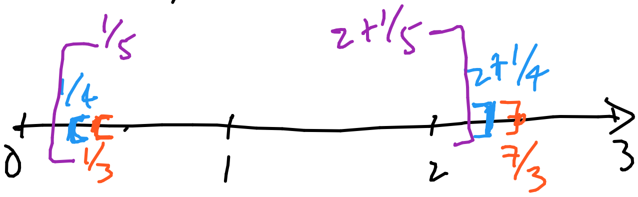
so $x > x$. $(\rightarrow \leftarrow)$

2.3.1K. Find the union and intersection of $\mathcal{A} := \{A_n : n \geq 3\}$
$A_n = \left[\frac{1}{n}, 2+\frac{1}{n}\right]$ for $n \in \mathbb{N} \sim \{1,2\}$.

Obs: $n=1$ $A_1 = [1,3]$, $n=2$ $A_2 = \left[\frac{1}{2}, \frac{5}{2}\right]$, $n=3$ $A_3 = \left[\frac{1}{3}, 7/3\right]$

$n=4$, $A_4 = \left[\frac{1}{4}, 2+\frac{1}{4}\right]$

$n=5$ $A_5 = \left[\frac{1}{5}, 2+\frac{1}{5}\right]$



Guess: (i) $\bigcup_{n \geq 3} A_n = (0, 7/3]$ and (ii) $\bigcap_{n\geq 3} A_n = [1/3, 2]$

Proof (i) "$\subseteq$" Pick $x \in \bigcup_{n\geq 3} A_n$, so $\exists n \in \mathbb{N}, n \geq 3$ s.t. $x \in A_n$
so $x \in \left[\frac{1}{n}, 2+\frac{1}{n}\right]$ for some $n \geq 3$, and $\left[\frac{1}{n}, 2+\frac{1}{n}\right] \subseteq (0, 7/3]$
$\forall n \geq 3$, sin $0 < \frac{1}{n}$ and $2+\frac{1}{n} \leq 7/3$ $\forall n \geq 3$.
Thus $x \in (0, 7/3]$.

"$\supseteq$" Pick $x \in \mathbb{R}$, $x \in (0, 7/3]$, so either $x \in [1, 7/3]$ or $x \in (0,1)$. Case 1: $x \in [1, 7/3] \subseteq A_3$ so $x \in A_3 \subseteq \bigcup_{n\geq 3} A_n$ ✓

Case 2: $x \in (0,1)$, in part. $x > 0$ so by APVII below
$\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} \leq x$. So if $n \geq 3 \Rightarrow x \in A_3 \subseteq \bigcup_{n\geq 3} A_n$ ✓
And if $n = 1$ or $2 \Rightarrow x \geq 1$ or $x \geq 2$ $(\rightarrow\leftarrow)$
In any case $x \in \bigcup_{n \geq 3} A_n$. #

Archimedean Property Version II (APVII): For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} \leq \varepsilon$.

By contradiction, assume $\sim(\forall \varepsilon > 0)(\exists n \in \mathbb{N})\left(\frac{1}{n} \leq \varepsilon\right)$ so
$(\exists \hat{\varepsilon} > 0)(\forall n \in \mathbb{N})\left(\frac{1}{n} > \hat{\varepsilon}\right)$ so $n < \underbrace{\frac{1}{\hat{\varepsilon}}}_{\in \mathbb{R}}$, $\forall n \in \mathbb{N}$ $(\rightarrow\leftarrow)$
Because $\mathbb{N}$ is not bounded from above.

Note that if $\varepsilon \in \mathbb{N}$ then AP VII coincides with our Arch. Prop seen in class for $a=1$, $b=\varepsilon$, $s=n$.

(ii) Let's show that $\bigcap_{n\geq 3} A_n = [1/3, 2]$. $A_n = [\frac{1}{n}, 2+\frac{1}{n}]$

"$\supseteq$" Since $\frac{1}{n} \leq \frac{1}{3}$ $\forall n \geq 3$, and $2 \leq 2+\frac{1}{n}$, $\forall n \geq 3$.

$\Rightarrow \underbrace{[\frac{1}{3}, 2]}_{B} \subseteq A_n$, $\forall n \geq 3$ so by Theorem 2.3.2(a)

we conclude that $B = [\frac{1}{3}, 2] \subseteq \bigcap_{n\geq 3} A_n$.

"$\subseteq$" Pick $x \in \bigcap_{n\geq 3} A_n$, so $x \in A_n = [\frac{1}{n}, 2+\frac{1}{n}]$, $\forall n \geq 3$

So $\frac{1}{n} \leq x \leq 2+\frac{1}{n}$, $\forall n \geq 3$. In particular ($n=3$)

$\frac{1}{3} \leq x$. Also $x \leq 2+\frac{1}{n}$, $\forall n \in \mathbb{N}$ (2 options):

1st Say taking $n \to \infty$ we get $x \leq 2$.
Calculus sequences $a_n \leq b_n$, $\forall n \Rightarrow \lim a_n \leq \lim b_n$. if lim exists.

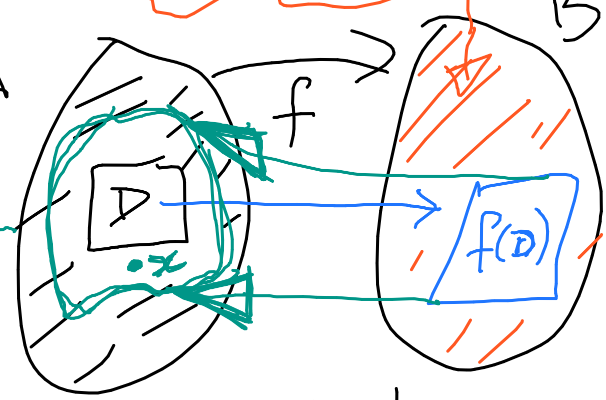
2nd: By contradiction, if $x > 2$, $\varepsilon = x - 2 > 0$, ...
use APUII to get $(\rightarrow\leftarrow)$ Thus $x \leq 2$.

Therefore $\frac{1}{3} \leq x \leq 2$, so $x \in [\frac{1}{3}, 2]$. #

4.5.10f. HINT: $f: A \to B$, $D \subseteq A$

w.t.s. $D = f^{-1}(f(D)) \iff f(A-D) \subseteq B - f(D)$

1st: $D \subseteq f^{-1}(f(D))$ always.
this is item (e).



($\Leftarrow$) w.t.s. $f^{-1}(f(D)) \subseteq D$

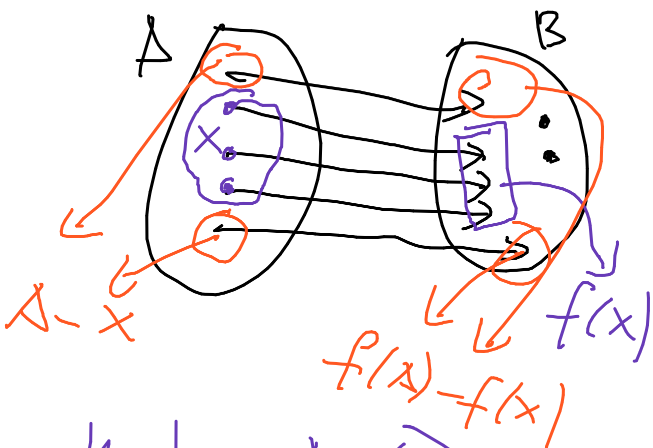
$x \in f^{-1}(f(D)) \iff f(x) \in f(D) \overset{?}{\Longrightarrow} \boxed{x \in D}$ w.t.s

In general $x \in D \Rightarrow f(x) \in f(D)$ BUT $\Leftarrow$ NOT in general. unless f is 1-1.

We know that $f^{-1}(f(D)) \subseteq A$ so $x \in A$.

Assume by contr. $x \notin D$. $\Rightarrow x \in A \wedge x \notin D \Rightarrow$
$x \in A - D \Rightarrow \dots$ FINISH IT!

4.5.13 $f: A \xrightarrow{1-1} B$, $X \subseteq A \Rightarrow f(A-X) = f(A) - f(X)$



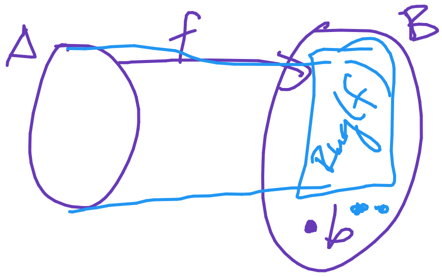
Proof: RHS $\subseteq$ LHS "always"

Pick $y \in B$. s.t. $y \in f(A) - f(X)$
$\Rightarrow y \in f(A)$ and $y \in f(X)^c$
$\Rightarrow y = f(a)$ for some $a \in A$.
<w.t.s. $b \in f(A-X) \Leftarrow a \notin X$, so that $a \in A - X$>

By contrad. Assume $a \in X \Rightarrow f(a) \in f(X) \Rightarrow y \in f(X)$ $(\to\leftarrow)$
so $a \notin X \Rightarrow a \in A - X$ and $y = f(a)$, that means $y \in f(A-X)$. <we did not use 1-1 here> //

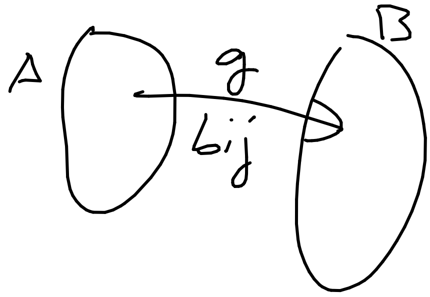
5.1.18(a) HINT: A, B finite $A \approx B$. $f: A \xrightarrow{1-1} B \Rightarrow f$ onto B.

Sketch: If f is not onto B, $\exists b \in B$ s.t. $b \notin \mathrm{Rng}(f)$.



$\exists \ell \in \mathbb{N}$, $\boxed{B \approx \mathbb{N}_\ell}$

On the other hand $A \approx B \Rightarrow \exists\ g: A \xrightarrow[\text{onto}]{1-1} B$



By Corollary 4.4.3 $f^{-1}: \mathrm{Rng}(f) \xrightarrow[\text{onto}]{1-1} A$ is a function

$g \circ f^{-1}: \mathrm{Rng}(f) \xrightarrow[\text{onto}]{1-1} B \xrightarrow{\varphi} \mathbb{N}_\ell$

$\exists h: \mathrm{Rng}(f) \xrightarrow[\text{onto}]{1-1} \mathbb{N}_\ell$ and show that

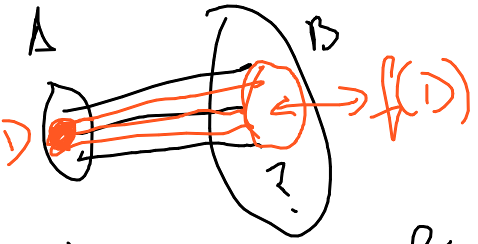
for $C = \mathrm{Rng}(f)$, since $b \notin C$ $\longrightarrow$ identify $\{b\}$ with $\{\ell\}$

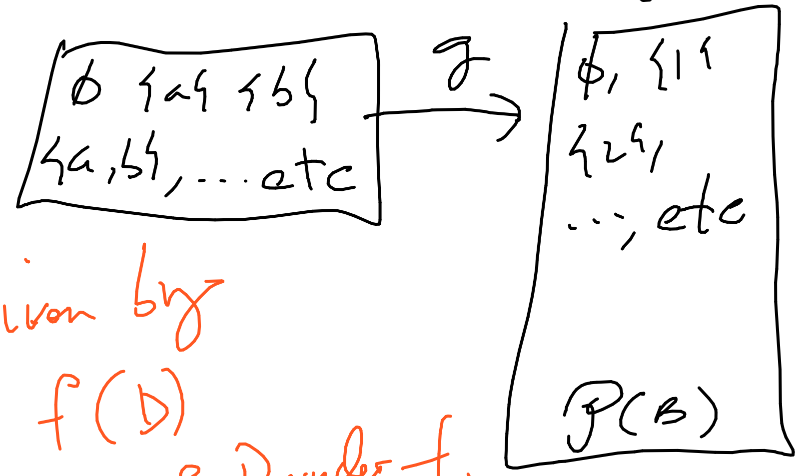
$\exists \tilde{f}: C \xrightarrow{1-1} \mathbb{N}_{\ell-1}$, so $\exists \tilde{h}: \mathbb{N}_\ell \xrightarrow{1-1} \mathbb{N}_{\ell-1}$ $(\rightarrow\leftarrow)$ by P.P.

$\tilde{h} = \tilde{f} \circ h^{-1}$

HINT FOR 5.4.5c: $\overline{\overline{A}} \le \overline{\overline{B}} \Rightarrow \overline{\overline{\mathcal{P}(A)}} \le \overline{\overline{\mathcal{P}(B)}}$

means $\exists f : A \xrightarrow{1-1} B$ w.t.s. $\exists g : \mathcal{P}(A) \xrightarrow{1-1} \mathcal{P}(B)$





Given $D \subseteq A$, $f(D) \subseteq B$

Claim $g : \mathcal{P}(A) \to \mathcal{P}(B)$ given by

$D \longmapsto g(D) = f(D)$

↑ image of D under f.

Use (and prove) f 1-1 $\Rightarrow \forall D \subseteq A$, $f^{-1}(f(D)) = D$.

HINT FOR 5.4.16c: $\mathcal{H} = \{ f \text{ s.t. } f : [0,1] \to [0,1] \}$

Question is $\overline{\overline{[0,1]}} \le \overline{\overline{\mathcal{H}}}$? or $=$ or $<$ or $>$ ...

Let's assume that we have proved items (a) and (b).

(a) $\forall \lambda : \underbrace{[0,1]}_{A} \to \underbrace{\mathcal{H}}_{B}$, λ is not bijective, so $\overline{\overline{A}} \ne \overline{\overline{B}}$, therefore $\overline{\overline{A}} < \overline{\overline{B}}$ or $\overline{\overline{B}} < \overline{\overline{A}}$.

(b) $\mathcal{H}$ is uncountable. ⟸ prove it.

(c) Consider the function $\varphi : [0,1] \to \mathcal{H}$, $a \longmapsto f_a : [0,1] \to [0,1]$

where $f_a(x) = a$, $\forall x \in [0,1]$.

So f_a is a constant function in $\mathcal{H}$.

Check that φ is 1-1! $f_a = f_b \Rightarrow a = b$

Thus, $\overline{\overline{A}} \le \overline{\overline{B}}$, therefore $\overline{\overline{A}} < \overline{\overline{\mathcal{H}}}$.

HINT FOR 5.4.8c: As cardinal numbers, $0 < 1 < 2 < \cdots < \aleph_0 < c$

Also, $\forall$ set A, $\bar{\bar{A}} < \overline{\overline{\mathcal{P}(A)}}$.

- $A = \mathbb{Q} \cup \{\pi\} \Rightarrow \bar{\bar{A}} = ?$ Show $A \approx \mathbb{N}$, so $\bar{\bar{A}} = \aleph_0$.
- $A = \mathbb{R} - \{\pi\} \Rightarrow A \approx \mathbb{R}$ $\xrightarrow[\pi]{\circ} \mathbb{R}$ clear $\bar{\bar{A}} \le \mathbf{c}$

  and $\mathbb{R} \approx (-\infty, \pi) \subseteq A$.
- $A = \mathcal{P}(\{0,1\}) \Rightarrow \bar{\bar{A}} = 4$.
- $A = \mathbb{R} - \mathbb{Z}$, clear $\bar{\bar{A}} \le \mathbf{c}$. similar.

  $\aleph_0 < \mathbf{c} = \mathbf{c}$

  $\overline{\overline{\mathbb{R} - \{\pi\}}} = \overline{\overline{\mathbb{R} - \mathbb{Z}}}$.
- Finally $\bar{\bar{\mathbb{R}}} < \overline{\overline{\mathcal{P}(\mathbb{R})}}$.

ON THEOREM 6.1.1(b).

Proof: Assume $*$ is associative with identity e. Suppose that b and d are inverses of a: $\underbrace{b*a}=a*b=e$, $d*a=\underbrace{a*d=e}$

$$\Rightarrow b = b*e = b*(a*d) \underset{a}{=} (b*a)*d = e*d = d.$$

We can't ensure this, if we don't have associativity #

HINT FOR 6.2.20: In $\mathbb{Z}_{20}$

(d) $x^2 = 9 \pmod{20}$. Find all x s.t. $\bar{x}\cdot\bar{x} = [x^2] = \bar{9}$.

(c) $x^2 = 0 \pmod{20}$ Want to find all classes $\bar{x} \in \mathbb{Z}_{20}$ s.t. $\bar{x}\cdot\bar{x} = [x^2] = \bar{0}$

$\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \ldots\ldots, \overline{19}\}$

- For $x = 0$, $\bar{x}\cdot\bar{x} = [0^2] = \bar{0}$. ✓
- For $x = 1$, $\bar{x}\cdot\bar{x} = [1^2] = \bar{1} \neq \bar{0}$ ✗

⋮

- For $x = 9$, $\bar{x}\cdot\bar{x} = [9^2] = [81] \neq \bar{0}$ since $20 \nmid 81-0$ ✗
- For $x = 10$, $\bar{x}\cdot\bar{x} = [10^2] = [100] = \overline{100} = \bar{0}$ since $20 \mid 100-0$. ✓

⋮ Test with all elements in $\mathbb{Z}_{20}$