Math 108, SS1 2020
July 3, 2020

**REVIEW HOMEWORK (sample)**

NAME(use CAPITAL letters, *first name first*): FULL NAME

NAME(sign): Signature

ID#: 123456789

**Instructions**: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Unless directed to do so, do *not* prove any theorem or proposition seen in class, and do not evaluate complicated expressions to give the result as a fraction or a decimal number. However, if you are using any of the problems in the textbook, then you have to solve or prove it.

**To deliver**: Submit your solutions on Canvas in either pdf or jpg format. Sign and submit this first page. Submit your self-video on Canvas as well, everything before 5:00PM.
Make sure that you have a total of 5 pages (including this one) with 4 problems.

| | |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| TOTAL | |

*Before starting*: Imagine that this is your real exam and solve it in less than 2 hours. Problems come with solutions, so solve them on a different paper. **Avoid reading the solutions**, and compare them with yours only when you finish your simulation.

1. For each of the following sets $A$ and relations $R$ on $A$, answer the following questions.
(a) $A = \{1,2,3,4\}$, $R = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,3),(4,4)\}$. ~~You do not need to justify Yes answers, but you do have to justify No answers.~~

Is $R$ reflexive? Yes.

6

Is $R$ symmetric? Yes.

Is $R$ transitive? No. $(3,1) \in R$ and $(1,2) \in R$, but $(3,2) \notin R$.

(b) $A = \mathbb{N} = \{1,2,3,\ldots\}$, $xRy$ if and only if $x+1 \le y$. ~~Justify all answers.~~ 2

Is $R$ reflexive? No. $(1,1) \notin R$, as $1+1 > 1$. 1

Is $R$ symmetric? No. $(1,2) \in R$ (as $1+1 \le 2$), but $(2,1) \notin R$ (as $2+1 > 1$). 3

Is $R$ transitive? Yes. If $x+1 \le y$ and $y+1 \le z$, then $x+1 \le z-1 \le z$.

(c) $A = \{1,2,3,4\}$, $R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}$. ~~You do not need to justify Yes answers, but you do have to justify No answers.~~

Is $R$ reflexive? Yes.

6

Is $R$ symmetric? Yes.

Is $R$ transitive? Yes.

**2.** Prove (by induction or any other correct method you know) that for every $n \in \mathbb{N}$, $10^n - 3^n$ is divisible by 7.

$(n=1)$ $10^1 - 3^1 = 7$, clearly divisible by 7.

$(n \to n+1)$ We need to show that $10^{n+1} - 3^{n+1}$ is divisible by 7. By I.H., $10^n - 3^n = 7k$, for an integer $k$. Therefore,

$$
\begin{aligned}
10^{n+1} - 3^{n+1} &= 10 \cdot 10^n - 3^{n+1} \\
&= 10 \cdot (7k + 3^n) - 3^{n+1} \\
&= 7 \cdot 10k + 10 \cdot 3^n - 3 \cdot 3^n \\
&= 7 \cdot 10k + 7 \cdot 3^n \\
&= 7(10k + 3^n),
\end{aligned}
$$

a multiple of 7. $\square$

3. Suppose $a$ and $b$ are elements of $\mathbb{N}$. Prove that

$$2a \leq b, \text{ and } b \leq a+3, \text{ and } b \text{ divides } 3a^2-1$$

if and only if

$$a = 1 \text{ and } b = 2.$$

(*Hint*. How do the two inequalities restrict $a$?)

($\Leftarrow$) Check : when $a=1$ and $b=2$,
$2 \cdot 1 \leq 2$, $2 \leq 1+3$, $2 \mid 3 \cdot 1^2 - 1$.

($\Rightarrow$) As $2a \leq b$ and $b \leq a+3$,
$2a \leq a+3$, $a \leq 3$. So we have 3
cases:

Case 1: $a=1$. Then $2 \leq b \leq 4$, and $b \mid 2$.
The only possibility is $b=2$.

Case 2: $a=2$. Then $4 \leq b \leq 5$ and $b \mid 11$.
Impossible as neither 4 nor 5
divide 11.

Case 3: $a=3$. Then $6 \leq b \leq 6$ and $b \mid 26$.
Impossible as 6 does not
divide 26.

Case one gives the only possibility: $a=1$, $b=2$. $\square$

4. Assume that $A$, $B$, $C$ are arbitrary subsets of $\mathbb{N}$. For each statement below, prove it or provide a counterexample.

(a) Prove: If $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

$$x \in A \cup C \Rightarrow (x \in A) \vee (x \in C)$$

$$(\text{as } A \subseteq B) \Rightarrow (x \in B) \vee (x \in C)$$

$$\Rightarrow x \in B \cup C$$

(b) Give a counterexample: Converse of (a). That is, $A \cup C \subseteq B \cup C \Rightarrow A \subseteq B$

Take $A = \mathbb{N}$, $B = \emptyset$, $C = \mathbb{N}$,

(c) Give a counterexample: If $A \cap B = \emptyset$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

Take $A = \{1\}$, $B = \{2\}$, $C = A$.

(d) Prove: If $1 \in A$, then $\{1\} \in \mathcal{P}(A)$.

If $1 \in A$, then $\{1\} \subseteq A$, and then $\{1\} \in \mathcal{P}(A)$.