Math 135A, Fall 2019
November 18, 2019

MIDTERM EXAM 2 (sample)

NAME(use CAPITAL letters, *first name first*): Full Name

NAME(sign): Signature

ID#: ID Number

SECTION: ______

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. You can use the Formula Sheet provided at our course website. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

Before starting: Imagine that this is your exam and solve it in 50 minutes. Problems 1 and 2 come with solutions, so solve them on a different paper. **Avoid reading the solutions**, and compare them with yours only when you finish your simulation.

1. A random variable X has density function

$$f(x) = \begin{cases} c(2x+1) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine c.

$$1 = c\int_0^1 (2x+1)\,dx = c\cdot(1 + 1)$$

$$\underline{\underline{c = \frac{1}{2}}}$$

(b) Compute Var(X).

$$EX = \frac{1}{2}\int_0^1 x\,(2x+1)\,dx = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2}\right) = \frac{7}{12}$$

$$E(X^2) = \frac{1}{2}\int_0^1 x^2(2x+1)\,dx = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{12}$$

$$\mathrm{Var}(X) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \underline{\underline{\frac{11}{144}}}$$

(c) Determine the probability density function of the random variable $Y = 1/\sqrt{X}$.

$$P(Y \le y) = P\left(\frac{1}{\sqrt{X}} \le y\right) = P\left(X \ge \frac{1}{y^2}\right)$$

$y \in (1, \infty)$

$$= \frac{1}{2}\int_{1/y^2}^{\infty} (2x+1)\,dx$$

$$f_Y(y) = \frac{d}{dy} P(Y \le y) = -\frac{1}{2}\cdot\left(2\cdot\frac{1}{y^2}+1\right)\cdot\left(-\frac{2}{y^3}\right)$$

$$= \frac{y^2+2}{y^5} \quad \text{for } y > 1$$

(0 otherwise)

$\boxed{2}$

Assume that Alice will arrive home this evening at a random time, uniformly distributed between 5 and 6 (all times p.m.). Bob promised to call Alice "sometime after 5," which in Bob's case means that he will, starting at 5, wait for an exponential amount of time with expectation 30 minutes and then call. Alice's arrival is independent of Bob's call.

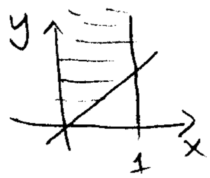
(a) Compute the probability that Alice will *not* miss Bob's call. Time unit = 1 hr

X = Alice's arrival is Uniform on $[0,1]$ (started at 5)

Y = Bob's call has density $f_Y(y) = \begin{cases} 2e^{-2y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$P(X \leq Y) = \int_0^1 dx \int_x^\infty 2e^{-2y}\,dy = \int_0^1 e^{-2x}\,dx = \underline{\underline{\frac{1}{2}(1-e^{-2})}}$$

Joint density $f(x,y) = \begin{cases} 2e^{-2y}, & y \geq 0,\ 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$



(b) Compute the conditional probability that, given that Alice will not miss the call. Bob will call before 6.

$$P(Y \leq 1 \mid X \leq Y) = \frac{P(Y \leq 1, X \leq Y)}{P(X \leq Y)}$$

$$= \frac{2}{1-e^{-2}} \int_0^1 dx \int_x^1 2e^{-2y}\,dy$$

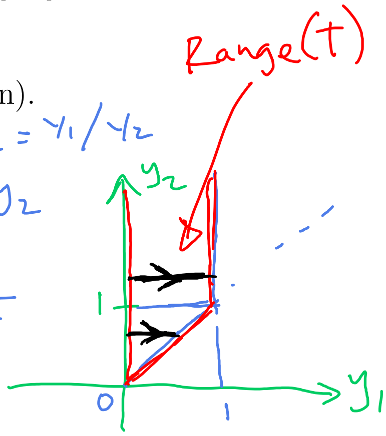
$$= \frac{2}{1-e^{-2}} \int_0^1 \left(-e^{-2} + e^{-2x}\right) dx$$

$$= \frac{2}{1-e^{-2}} \left(-e^{-2} + \frac{1}{2} - \frac{1}{2}e^{-2}\right)$$

$$= \frac{1-3e^{-2}}{1-e^{-2}}$$

3. Assume that X_1 and X_2 are independent and identically distributed Unif[0,1]. Let $Y_1 = X_1$ and $Y_2 = X_1/X_2$

(a) [10pts] Compute the joint density of Y_1, Y_2 (Be careful with the domain).



$Y_1 = X_1$, $Y_2 = X_1/X_2 \Rightarrow X_1 = Y_1$ and $X_2 = X_1/Y_2$, so $X_2 = Y_1/Y_2$

Domain: $0 < y_1 < 1$ and $0 < y_1/y_2 < 1$, so $0 < y_1 < y_2$

$$J(y_1,y_2) = \frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2}\cdot\frac{\partial x_2}{\partial y_1} = 1\times\left(-\frac{y_1}{y_2^2}\right) - 0\times\frac{1}{y_2} = \frac{-y_1}{y_2^2}$$

Since X_1, X_2 are independent, $f_{X_1,X_2}(x_1,x_2) = 1$ for $0 < x_1 < 1$, $0 < x_2 < 1$. Thus, by the change of variables

$$f_{Y_1,Y_2}(y_1,y_2) = 1\times|J(y_1,y_2)| = y_1/y_2^2 \text{ if } 0<y_1<1,\ 0<y_1<y_2<\infty,$$
(0 otherwise)

(b) [7pts] Compute $\mathbb{P}(Y_2 \le 3Y_1)$.

$$\mathbb{P}(Y_2 \le 3Y_1) = \mathbb{P}\left(\frac{X_1}{X_2} \le 3X_1\right) = \mathbb{P}\left(\frac{1}{3} \le X_2\right) = 1 - F_{X_2}(1/3)$$
$$= 1 - 1/3 = 2/3.$$

(c) [8pts] Find the marginal densities f_{Y_1} and f_{Y_2}.

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_{y_1}^{\infty}\frac{y_1}{y_2^2}\,dy_2 = 1, \text{ for } 0<y_1<1$$

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_1 = \begin{cases}\int_0^{y_2}\frac{y_1}{y_2^2}\,dy_1, & \text{if } 0<y_2<1\\ \int_0^1\frac{y_1}{y_2^2}\,dy_1, & \text{if } y_2\ge 1\end{cases}$$

see the picture

$$= \begin{cases}\frac{1}{y_2^2}\frac{y_1^2}{2}\Big|_0^{y_2} & \text{if } ''\\ \frac{1}{y_2^2}\frac{y_1^2}{2}\Big|_0^1 & \text{if } ''\end{cases} = \begin{cases}\frac{1}{2}, & \text{if } 0<y_2<1\\ \frac{1}{2y_2^2}, & \text{if } y_2\ge 1.\end{cases}$$

4. Determine whether the following statements are True or False. Justify your answer with a proof or a counterexample as appropriate.

(a) [7pts]. If X, Y are independent Poisson variables with respective parameters λ and μ, then the conditional distribution of $X \big| X+Y=n$ is $\text{Bin}(n, \lambda/\mu)$.

False: If $\lambda > \mu$, then $\text{Bin}(n, \lambda/\mu)$ doesn't make sense.

(b) [6pts]. If $X \sim \text{Beta}(a,b)$, where $a, b > 0$, then $\mathbb{E}(X) = \dfrac{a+b}{a}$.

False: Since $0 \leq X \leq 1$, then $0 \leq \mathbb{E}X \leq 1$. But $\frac{a+b}{a} > 1$.
Indeed you should know that $\mathbb{E}X = \dfrac{a}{a+b}$

(c) [6pts]. If $\alpha, \beta > 0$ and $X \sim \text{Exp}(\alpha)$, then $Y = X^{1/\beta}$ has the Weibull distribution with parameters α, β.

True: $F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(X^{1/\beta} \leq x) = \mathbb{P}(X \leq x^\beta)$
$= 1 - e^{-\lambda x^\beta}$, for $x \geq 0$.

(d) [6pts]. If X, Y have joint density given by $f(x,y) = \dfrac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\dfrac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$, for some constant $\rho \in (-1,1)$, then $\mathbb{E}(Y|X) = \rho X$.

True: This is example 4.6 (7) in the book (Read it!)