Problem 2

(a)

The sample space Ω consists of all permutations of $\{n_1, n_2, n_3, n_4\}$. Thus, four players will have in total $|\Omega| = 4!$ assignments for the numbers. For instance, $X(n_2n_3n_1n_4) = 0$, $X(n_3n_1n_4n_2) = 1$, $X(n_3n_1n_2n_4) = 2$ and $X(n_4n_2n_3n_1) = 3$.

1. Consider player 1 never wins, so the player 2 should always choose a number bigger than player 1:
$$\mathbb{P}(X = 0) = \frac{\binom{4}{2} \times 2!}{4!} = \frac{12}{24}$$
2. In this case, player 1 will have number larger than player 2 but smaller than player 3, so for player 1,2,3 we can have
$$\mathbb{P}(X = 1) = \frac{\binom{4}{3}}{4!} = \frac{4}{24}$$
3. Since we know that player 1 only win twice but he lost in the final round, which means he must choose n_3 and players 2 and 3 choose numbers n_1 or n_2:
$$\mathbb{P}(X = 2) = \frac{2!}{4!}$$
4. Consider player win till the final round, we must have player 1 assigned n_4, so we will have 3! arrangements for the remaining three players
$$\mathbb{P}(X = 3) = \frac{3!}{4!}$$

You should always check that your probability add up to 1

(b)

By using the formula for expectation, we will have:

$$\mathbb{E}(X) = \sum_{i=0}^{3} i \times \mathbb{P}(X = i) = 0 \times \frac{12}{24} + 1 \times \frac{4}{24} + 2 \times \frac{2}{24} + 3 \times \frac{6}{24} = \frac{26}{24}$$

(c)

We know that $F_X(x) = \mathbb{P}(X \leq x)$, so

$$F(x) = \begin{cases} 0, \ x < 0 \\ \frac{12}{24}, 0 \leq x < 1 \\ \frac{16}{24}, 1 \leq x < 2 \\ \frac{18}{24}, 2 \leq x < 3 \\ 1, \ x \geq 3 \end{cases}$$

Now, you can draw the distribution function.

Problem 4

(a)

No. For example, suppose that we toss 1 fair coin. Let A be the event that the coin lands head and let B be the event that the coin lands tail. Then $A \cap B = \emptyset$, but A and B are NOT independent, since $P(A \cap B) = 0 \neq P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2}$.

(b)

Yes. $\mathbb{P}(X < 7) = 1$ implies $\mathbb{P}(X \geq 7) = 0$, so $0 \leq \mathbb{P}(X > 9) \leq \mathbb{P}(X \geq 7) = 0$.

(c)

No. Unless $p = 1/2$, since in general $Y \sim \text{Ber}(1-p)$ [prove it].

(d)

Yes. By definition of conditional probability, we have for $i, k \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{P}(X = k + i \mid X > i) \\
=&\frac{\mathbb{P}(\{X = k + i\} \cap \{X > i\})}{\mathbb{P}(X > i)} \\
=&\frac{\mathbb{P}(X = k + i)}{\mathbb{P}(X > i)} = \frac{(1-p)^{k+i-1}p}{(1-p)^i} = (1-p)^{k-1}p = \mathbb{P}(X = k)
\end{aligned}$$

This is called the memoryless property of the geometric distribution.