Math 135A, Fall 2019
November 25, 2019

MIDTERM EXAM 2

NAME(use CAPITAL letters, *first name first*):____

NAME(sign):________________________________

SECTION:_________________________

ID#:_______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. You can use the Formula Sheet provided at our course website. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.
You may write on *both* sides of your paper.

1	
2	
3	
4	
TOTAL	

1. Assume that X has density function $f(x) = c|x|$, for $x \in (-3, 3)$ [and 0 otherwise]. Find

(a) [8pts] the constant c.

$$\frac{1}{c} = \int_{-3}^{3} |x|\,dx = 2\int_0^3 x\,dx = x^2\Big|_0^3 = 9 \implies c = \frac{1}{9}$$

(b) [8pts] the variance of the random variable $Y = X + 3$.

$$\mathbb{E}(X) = c\int_{-3}^{3} x|x|\,dx = 0 \quad \text{[odd function]}$$

$$\mathbb{E}(X^2) = c\int_{-3}^{3} x^2|x|\,dx = 2c\int_0^3 x^3\,dx = \frac{c}{2}x^4\Big|_0^3 = \frac{81}{2}c$$

thus, $\mathrm{Var}(Y) = \mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \frac{81c}{2} - 0 = \frac{9}{2}$

(c) [9pts] the density function of the random variable $Z = X^2$. For $z \in (0, 9)$:

$$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}(X^2 \le z) = \mathbb{P}(|X| \le \sqrt{z}) = \mathbb{P}(-\sqrt{z} \le X \le \sqrt{z})$$

$$= \int_{-\sqrt{z}}^{\sqrt{z}} c|x|\,dx = 2c\int_0^{\sqrt{z}} x\,dx = c\,x^2\Big|_0^{\sqrt{z}} = cz,$$

so, $f_Z(z) = F_Z'(z) = c = \frac{1}{9}$, for $0 < z < 9$,

and 0 otherwise.

2. You arrive at a bus stop at 9 o'clock, knowing that the bus will arrive at some time uniformly distributed between 9 and 9:30.

   (a) [10pts] What is the probability that you will have to wait longer than 5 minutes?

   (b) [15pts] If at 9:10 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

3. Assume that X_1 and X_2 are independent and identically distributed $\mathcal{N}(0,1)$. Consider $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

(a) [10pts] Compute the joint density of Y_1, Y_2. Are Y_1 and Y_2 independent?

$Y_1 + Y_2 = 2X_1 \Rightarrow X_1 = (Y_1+Y_2)/2$. $\quad Y_1 - Y_2 = 2X_2 \Rightarrow X_2 = \frac{Y_1 - Y_2}{2}$.

$-\infty < x_1, x_2 < \infty \iff -\infty < y_1+y_2,\ y_1-y_2 < \infty \iff -\infty < y_1, y_2 < \infty$.

$$J(y_1,y_2) = \frac{\partial x_1}{\partial y_1}\cdot\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1} = \frac{1}{2}\cdot\left(-\frac{1}{2}\right) - \frac{1}{2}\cdot\frac{1}{2} = -\frac{1}{2}.$$

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_1^2}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x_2^2}\cdot|J(y_1,y_2)| = \frac{1}{2\pi}e^{-\frac{1}{2}\left(\left(\frac{y_1+y_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2\right)}\cdot\frac{1}{2}$$

$$= \frac{1}{4\pi}e^{-\frac{1}{2}\left(2\frac{y_1^2}{4}+2\frac{y_2^2}{4}\right)} = \underbrace{\frac{1}{\sqrt{2\pi\cdot 2}}e^{-\frac{1}{2}\frac{y_1^2}{2}}}_{f_Z(y_1)}\cdot\underbrace{\frac{1}{\sqrt{2\pi\cdot 2}}e^{-\frac{1}{2}\frac{y_2^2}{2}}}_{f_Z(y_2)},$$

where $Z \sim \mathcal{N}(0,2)$

Thus, Y_1, Y_2 are independent.

(b) [7pts] Compute $\mathbb{P}(Y_1 \le Y_2)$.

$$\mathbb{P}(X_1+X_2 \le X_1 - X_2) = \mathbb{P}(2X_2 \le 0)$$
$$= \mathbb{P}(X_2 \le 0) = 1/2, \text{ since } X_2 \sim \mathcal{N}(0,1).$$

(c) [8pts] Find f_{Y_1}, $\mathrm{Cov}(Y_1,Y_2)$ and $\mathbb{E}(Y_1|Y_2)$.

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_{-\infty}^{\infty} f_Z(y_1)\cdot f_Z(y_2)\,dy_2$$
$$= f_Z(y_1)\int_{-\infty}^{\infty} f_Z(y_2)\,dy_2 = f_Z(y_1), \text{ with } Z\sim\mathcal{N}(0,2)$$

Y_1 and Y_2 are independent, thus

$\mathrm{Cov}(Y_1,Y_2) = \mathbb{E}(Y_1Y_2) - \mathbb{E}(Y_1)\mathbb{E}(Y_2) = 0$, and

$\mathbb{E}(Y_1|Y_2) = \mathbb{E}(Y_1) = 0$.

4. Determine whether the following statements are True or False. Justify your answer with a proof or a counterexample as appropriate.

(a) [8pts] If X and Y are independent Ber(1/2), then $X|(X+Y=1) \sim \text{Ber}(1/2)$.

(b) [8pts]. If $X \sim Exp(\lambda)$, then $\mathbb{P}(X > t+s | X > t) = \mathbb{P}(X > s)$ for all $s, t > 0$.

(c) [9pts]. If $X \sim Beta(1,1)$, then $\mathbb{E}(X^n) = \dfrac{1}{(n+1)!}$ for all $n \in \mathbb{N}$.