Homework 1: Some partial solutions

Most of the following solutions are not complete, they reveal the main idea, but you have to complete all details. If you spot any mistake in computations, please let me know and you will earn a bonus...

Problem 2

$$a \to (ii), d \to (i), b \to (iii), c \to (iv)$$

Problem 3

By induction and inclusion exclusion principle, we can show it for finite unions:

$$\begin{aligned} \mathbb{P}(A_1) &\le \mathbb{P}(A_1), \text{ and assuming it holds for } n \\ \mathbb{P}(\bigcup_{i=1}^{n+1} A_i) &\le \mathbb{P}(\bigcup_{i=1}^{n} A_i) + \mathbb{P}(A_{n+1}) - \mathbb{P}(\bigcup_{i=1}^{n} A_n \cap A_{n+1}) \\ &\le \sum_{i=1}^{n+1} \mathbb{P}(A_i) \end{aligned}$$

How to deduce it for infinite unions?

Problem 4

We have to apply sandwich theorem from calculus.

$$\begin{aligned} \mathbb{P}(A_n \cap B_n) = \mathbb{P}(A_n) + \mathbb{P}(B_n) - \mathbb{P}(A_n \cup B_n) \ge \mathbb{P}(A_n) + \mathbb{P}(B_n) - 1, \text{ so} \\ \mathbb{P}(A_n) + \mathbb{P}(B_n) - 1 \le \mathbb{P}(A_n \cap B_n) \le \mathbb{P}(B_n) \end{aligned}$$

Finish it.

Problem 5

Define

$$C_n = \bigcup_{k=n}^{\infty} A_k, B_n = \bigcap_{k=n}^{\infty} A_k,$$

and note that $B_n \subset A_n \subset C_n$. Now, since $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = B$, then $\lim_n C_n = B = \lim_n B_n$, so by the continuity of the probability we obtain

$$\mathbb{P}(B_n) \to \mathbb{P}(B), \text{ and } \mathbb{P}(C_n) \to \mathbb{P}(B).$$

Therefore $\mathbb{P}(B) \le \lim_{n\to\infty} \mathbb{P}(A_n) \le \mathbb{P}(B)$, where we used sandwich theorem again.

Problem 6

In order to set a suitable Ω, let us consider the possible outcomes of the experiment. Maybe it stops after n stages, in this case all possible outcomes belong to $\{1, 2, 3, 4, 5\}^{n-1} \times \{6\}$, why?
But, maybe the experiment does not stop (for instance, if you get an infinite sequence of 5s), in this case all possible outcomes belong to $\{1, 2, 3, 4, 5\}^{\mathbb{N}}$, why?, so

$$\Omega = \bigcup_{n=1}^{\infty} (\{1, 2, 3, 4, 5\}^{n-1} \times \{6\}) \cup \{1, 2, 3, 4, 5\}^{\mathbb{N}}.$$

For the second question, observe that $E_n = \{1,2,3,4,5\}^{n-1} \times \{6\}$, that is, the points of Ω which are in E_n are exactly the vectors of the form $(x_1, x_2, \ldots, x_{n-1}, 6)$, with $1 \le x_i \le 5,\ \forall 1 \le i \le n-1$..
The final question asks us to consider $(\cup_1^\infty E_n)^c$. The union inside represents the occurence of some E_n, that is, the event that the experiment stops after n stages, for some n. The complement then is the event that a 6 never appears, which corresponds to $\{1,2,3,4,5\}^{\mathbb{N}}$.

Problem 7

Once you place a rook, you eliminate its row and column, so

$$\mathbb{P}(A) = \frac{(8!)^2}{64 \times 63 \times 62 \times ... \times 57} \approx 9.109 \times 10^{-6}$$

Problem 8

By symmetry between the positions, this probability does not depend on i, so assume $i = 1$. We choose one girl from g available ways, we then arrange the remaining $(b+g-1)$ people in the $(b+g-1)$ positions. The total number of different arrangement is $(b+g)!$. So the probability is:

$$\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$$

Problem 9

In each of the n stages, each outcome (i,k) has probability $\frac{1}{36}$. Consider the complementary event that $(6,6)$ never appears, this has probability $(35/36)^n$, so the desired probability is

$$1 - \left(\frac{35}{36}\right)^n \ge \frac{1}{2}, \text{ for } n \ge ?$$

Problem 10

Consider the event $E_i = \{i\text{-th couple sits together}\}$ for $i = 1,2,3,4$, and explain the following computations

$$\begin{aligned}
\mathbb{P}(E_i) &= \frac{2 \cdot 7!}{8!} = \frac{1}{4} \\
\mathbb{P}(E_i \cap E_j) &= \frac{2^2 \cdot 6!}{8!} = \frac{1}{14} \\
\mathbb{P}(E_i \cap E_j \cap E_k) &= \frac{2^3 \cdot 5!}{8!} = \frac{1}{42} \\
\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) &= \frac{2^4 \times 4!}{8!} = \frac{1}{105}.
\end{aligned}$$

Thus, by the Inclusion-Exclusion Principle and symmetry between the couples:

$$\mathbb{P}\left(\bigcup_{i=1}^4 E_i\right) = \binom{4}{1}\mathbb{P}(E_1) - \binom{4}{2}\mathbb{P}(E_1 \cap E_2) + \binom{4}{3}\mathbb{P}(E_1 \cap E_2 \cap E_3) - \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) = 1 - \frac{12}{35},$$

and the desired probability is $\mathbb{P}(\bigcap_{i=1}^4 E_i^c) = \frac{12}{35}$.

1 Solutions for Section 1.2

Problem 1

(a) Let $a \in (\cup A_i)^{\mathrm{c}}$. Then $a \notin \cup A_i$, so that $a \in A_i^{\mathrm{c}}$ for all i. Hence $(\cup A_i)^{\mathrm{c}} \subseteq \cap A_i^{\mathrm{c}}$. Conversely, if $a \in \cap A_i^{\mathrm{c}}$, then $a \notin A_i$ for every i. Hence $a \notin \cup A_i$, and so $\cap A_i^{\mathrm{c}} \subseteq (\cup A_i)^{\mathrm{c}}$. The first De Morgan law follows.
(b) Applying part (a) to the family $\{A_i^{\mathrm{c}} : i \in I\}$, we obtain that $(\cup_i A_i^{\mathrm{c}})^{\mathrm{c}} = \bigcap_i (A_i^{\mathrm{c}})^{\mathrm{c}} = \bigcap_i A_i$ Taking the complement of each side yields the second law.

Problem 2

(i) $A \cap B = (A^{\mathrm{c}} \cup B^{\mathrm{c}})^{\mathrm{c}}$
(ii) $A \backslash B = A \cap B^{\mathrm{c}} = (A^{\mathrm{c}} \cup B)^{\mathrm{c}}$
(iii) $A \Delta B = (A \backslash B) \cup (B \backslash A) = (A^{\mathrm{c}} \cup B)^{\mathrm{c}} \cup (A \cup B^{\mathrm{c}})^{\mathrm{c}}$
Now $\mathcal{F}$ is closed under the operations of countable unions and complements, and therefore each of these sets lies in $\mathcal{F}$.

Problem 3

Let us number the players $1, 2, \ldots, 2^n$ in the order in which they appear in the initial table of draws. The set of victors in the first round is a point in the space $V_n = \{1,2\} \times \{3,4\} \times \cdots \times \{2^n - 1, 2^n\}$. Renumbering these victors in the same way as done for the initial draw, the set of second-round victors can be thought of as a point in the space V_{n-1}, and so on. The sample space of all possible outcomes of the tournament may therefore be taken to be $V_n \times V_{n-1} \times \cdots \times V_1$, a set containing $2^{2^{n-1}} 2^{2^{n-2}} \cdots 2^1 = 2^{2^n - 1}$ points.

Problem 4

4. We must check that $\mathcal{G}$ satisfies the definition of a σ -field:
(a) $\varnothing \in \mathcal{F}$, and therefore $\varnothing = \varnothing \cap B \in \mathcal{G}$,
(b) if $A_1, A_2, \ldots \in \mathcal{F}$, then $\cup_i (A_i \cap B) = (\cup_i A_i) \cap B \in \mathcal{G}$
(c) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ so that $B \backslash (A \cap B) = A^c \cap B \in \mathcal{G}$.
Note that $\mathcal{G}$ is a σ -field of subsets of B but not a σ -field of subsets of Ω, since $C \in \mathcal{G}$ does not imply that $C^c = \Omega \backslash C \in \mathcal{G}$.

Problem 5

(a), (b), and (d) are identically true; (c) is true if and only if $A \subseteq C$

2 Solutions for Section 1.3

Problem 1

(i) We have (using the fact that $\mathbb{P}$ is a non-decreasing set function) that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 = \frac{1}{12}$$

Also, since $A \cap B \subseteq A$ and $A \cap B \subseteq B, \mathbb{P}(A \cap B) \leq \min\{\mathbb{P}(A), \mathbb{P}(B)\} = \frac{1}{3}$

These bounds are attained in the following example. Pick a number at random from $\{1, 2, \ldots, 12\}$. Taking $A = \{1, 2, \ldots, 9\}$ and $B = \{9, 10, 11, 12\}$, we find that $A \cap B = \{9\}$, and so $\mathbb{P}(A) = \frac{3}{4}$, $\mathbb{P}(B) = \frac{1}{3}, \mathbb{P}(A \cap B) = \frac{1}{12}$. To attain the upper bound for $\mathbb{P}(A \cap B)$, take $A = \{1, 2, \ldots, 9\}$ and $B = \{1, 2, 3, 4\}$.

(ii) Likewise we have in this case $\mathbb{P}(A \cup B) \leq \min\{\mathbb{P}(A) + \mathbb{P}(B), 1\} = 1$, and $\mathbb{P}(A \cup B) \geq \max\{\mathbb{P}(A), \mathbb{P}(B)\} = \frac{3}{4}$. These bounds are attained in the examples above.

Problem 2

(i) We have (using the continuity property of $\mathbb{P}$) that

$$\mathbb{P}(\text{ no head ever }) = \lim_{n\to\infty} \mathbb{P}(\text{ no head in first } n \text{ tosses }) = \lim_{n\to\infty} 2^{-n} = 0$$

so that $\mathbb{P}(\text{ some head turns up }) = 1 - \mathbb{P}(\text{ no head ever }) = 1$

(ii) Given a fixed sequence s of heads and tails of length k, we consider the sequence of tosses arranged in disjoint groups of consecutive outcomes, each group being of length k. There is probability 2^{-k} that any given one of these is s, independently of the others. The event fone of the first n such groups is $s\}$ is a subset of the event $\{s$ occurs in the first nk tosses $\}$. Hence (using the general properties of probability measures) we have that:

$$\begin{aligned}
\mathbb{P}(s \text{ turns up eventually}) &= \lim_{n\to\infty} \mathbb{P}(s \text{ occurs in the first } nk \text{ tosses}) \\
&\geq \lim_{n\to\infty} \mathbb{P}(s \text{ occurs as one of the first } n \text{ groups }) \\
&= 1 - \lim_{n\to\infty} \mathbb{P}(\text{ none of the first } n \text{ groups is } s) \\
&= 1 - \lim_{n\to\infty} \left(1 - 2^{-k}\right)^n = 1
\end{aligned}$$

Problem 3

Lay out the saucers in order, say as RRWWSS. The cups may be arranged in 6! ways, but since each pair of a given colour may be switched without changing the appearance, there are $6! \div (2!)^3 = 90$ distinct arrangements. By assumption these are equally likely. In how many such arrangements is no cup on a saucer of the same colour? The only acceptable arrangements in which cups of the same colour are paired off are WWSSRR and SSRRW; by inspection, there are a further eight arrangements in which the first pair of cups is either SW or WS, the second pair is either RS or SR, and the third either RW or WR. Hence the required probability is $10/90 = \frac{1}{9}$.

Problem 4

We prove this by induction on n , considering first the case $n = 2$. Certainly $B = (A \cap B) \cup (B \backslash A)$ is a union of disjoint sets, so that $\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(B \backslash A)$. Similarly $A \cup B = A \cup (B \backslash A)$, and so:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \backslash A) = \mathbb{P}(A) + \{\mathbb{P}(B) - \mathbb{P}(A \cap B)\}$$

Hence the result is true for $n = 2$. Let $m \geq 2$ and suppose that the result is true for $n \leq m$. Then it is true for pairs of events, so that:

$$\begin{aligned}
\mathbb{P}\left(\bigcup_1^{m+1} A_i\right) &= \mathbb{P}\left(\bigcup_1^{m} A_i\right) + \mathbb{P}\left(A_{m+1}\right) - \mathbb{P}\left\{\left(\bigcup_1^{m} A_i\right) \cap A_{m+1}\right\} \\
&= \mathbb{P}\left(\bigcup_1^{m} A_i\right) + \mathbb{P}\left(A_{m+1}\right) - \mathbb{P}\left\{\bigcup_1^{m} \left(A_i \cap A_{m+1}\right)\right\}
\end{aligned}$$

Using the induction hypothesis, we may expand the two relevant terms on the right-hand side to obtain the result.

Let A_1, A_2, and A_3 be the respective events that you fail to obtain the ultimate, penultimate, and ante-penultimate Vice-Chancellors. Then the required probability is, by symmetry.

$$\begin{aligned}
1 - \mathbb{P}\left(\bigcup_1^{3} A_i\right) &= 1 - 3\mathbb{P}\left(A_1\right) + 3\mathbb{P}\left(A_1 \cap A_2\right) - \mathbb{P}\left(A_1 \cap A_2 \cap A_3\right) \\
&= 1 - 3\left(\frac{4}{5}\right)^6 + 3\left(\frac{3}{5}\right)^6 - \left(\frac{2}{5}\right)^6
\end{aligned}$$

Problem 5

$$\mathbb{P}\left(\bigcap_{r=1}^{\infty} A_r\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{r=1}^{n} A_r\right) = \lim_{n\to\infty}\left[1 - \mathbb{P}\left(\left(\bigcap_{r=1}^{n} A_r\right)^{\mathrm{c}}\right)\right]$$
$$= 1 - \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{r=1}^{n} A_r^{\mathrm{c}}\right) \geq 1 - \lim_{n\to\infty} \sum_{r=1}^{n} \mathbb{P}\left(A_r^{\mathrm{c}}\right) = 1$$

Problem 6

We have that $1 = \mathbb{P}\left(\bigcup_1 A_r\right) = \sum_r \mathbb{P}\left(A_r\right) - \sum_{r<s} \mathbb{P}\left(A_r \cap A_s\right) = np - \frac{1}{2}n(n-1)q$. Hence $p \geq n^{-1}$, and $\frac{1}{2}n(n-1)q = np - 1 \leq n - 1$

Problem 7

$$1 = \mathbb{P}\left(\bigcup_1^n A_r\right) = \sum_r \mathbb{P}\left(A_r\right) - \sum_{r<s} \mathbb{P}\left(A_r \cap A_s\right) + \sum_{r<s<t} \mathbb{P}\left(A_r \cap A_s \cap A_t\right)$$
$$= np - \binom{n}{2} q + \binom{n}{3} x$$

since at least two of the events occur with probability $\frac{1}{2}$

$$\frac{1}{2} = \mathbb{P}\left(\bigcup_{r<s} \left(A_r \cap A_s\right)\right) = \sum_{r<s} \mathbb{P}\left(A_r \cap A_s\right) - \frac{1}{2} \sum_{r<s, t<u, (r,s)\neq(t,u)} \mathbb{P}\left(A_r \cap A_s \cap A_t \cap A_u\right)$$

By a careful consideration of the first three terms in the latter series, we find that:

$$\frac{1}{2} = \binom{n}{2} q - 3\binom{n}{3} x + \binom{n}{3} x$$

Hence $\frac{3}{2} = np - \binom{n}{3} x$, so that $p \geq 3/(2n)$. Also, $\binom{n}{2} q = 2np - \frac{5}{2}$, whence $q \leq 4/n$