Homework 2: Some partial solutions

Problem 3

Let $B, C, A_1 A_2 A_3, \ldots$ be events. Prove that:

1. If $A_1, A_2, \ldots$ are pairwise disjoint $, \mathbb{P}(A_n) > 0$ and $\mathbb{P}(B|A_n) \geq p$ for all $n \in N$, then $\mathbb{P}(B| \cup A_n) \geqslant p$
2. If $A_1, A_2, ...$ are decreasing, and $\mathbb{P}(A_{n+1} \mid A_n) \leq \frac{1}{2} \quad \forall n \in \mathbb{N}$, then $\mathbb{P}(A_n) \longrightarrow 0$ as $n \longrightarrow \infty$.
3. If $A_1, A_2, ...$ are pairwise disjoint and $\mathbb{P}(B \mid A_n) = \mathbb{P}(C \mid A_n)$ $\forall n \in \mathbb{N}$, then $\mathbb{P}(B \mid \bigcup_n A_n) = \mathbb{P}(C \mid \bigcup_n A_n)$
4. If $A_1, A_2, ...$ is a partition of Ω then $\mathbb{P}(B|C) = \sum_n \mathbb{P}(A_n \mid C)\mathbb{P}(B \mid A_n \bigcap C)$

Problem 5

Let $B, A_1, A_2, ...$ be events that has positive probability such that $A_1, A_2, A_3, ...$ is a partition of Ω, prove that:

$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(B \mid A_i)\mathbb{P}(A_i)}{\sum_n \mathbb{P}(B \mid A_n)\mathbb{P}(A_n)}$$

Problem 6

Toss 2 fair dice. Given that the result shows different numbers, what is the conditional probability that:

1. At least one die shows 6
2. the sum of the numbers is 8

Problem 7

In a multiple choice test, with m choices, the probability of a student knowing the answer is p. If she knows the answer then she chooses the right answer with probability 1. But, if the doesn't then she chooses the night answer with probability $\frac{1}{m}$.What is the probability that she knew the answer given that she choose the right answer? Compute the limit of this probability as:
(i) $m \to \infty, \quad p$ fixed. (ii) $P \to 0, m$ fixed.

Problem 3

1. $$\mathbb{P}(B \mid \cup A_n) = \frac{\mathbb{P}(\cup A_n \mid B)\mathbb{P}(B)}{\mathbb{P}(\cup A_n)} = \frac{\sum_n \mathbb{P}(A_n \mid B)\mathbb{P}(B)}{\mathbb{P}(\cup A_n)} = \frac{\sum_n \mathbb{P}(B \mid A_n)\mathbb{P}(A_n)}{\sum_n \mathbb{P}(A_n)} \geq p$$

2. Using the Law of total probability, we will have:

$$\mathbb{P}(A_{n+1}) = \mathbb{P}(A_{n+1} \mid A_n)\mathbb{P}(A_n) + \mathbb{P}(A_{n+1} \mid A_n^c)\mathbb{P}(A_n^c) = \mathbb{P}(A_{n+1} \mid A_n)\mathbb{P}(A_n) \leq \frac{1}{2}\mathbb{P}(A_n)$$

Since we know that

$$\mathbb{P}(A_{n+1}|A_n^c) = \frac{\mathbb{P}(A_{n+1} \cap A_n^c)}{\mathbb{P}(A_n^c)} = 0$$

Thus, for all $n \in \mathbb{N}$:

$$0 \leq \mathbb{P}(A_{n+1}) \leq \left(\frac{1}{2}\right)^n \mathbb{P}(A_1)$$

3. Using the computations in item 1

$$\mathbb{P}(B \mid \cup A_n) = \frac{\sum_n \mathbb{P}(B \mid A_n)\mathbb{P}(A_n)}{\sum_n \mathbb{P}(A_n)} = \frac{\sum_n \mathbb{P}(C \mid A_n)\mathbb{P}(A_n)}{\sum_n \mathbb{P}(A_n)} = \mathbb{P}(C \mid \cup A_n)$$

4. Start with RHS and get to LHS.

Problem 5

$$\mathbb{P}\left(A_j|B\right) = \frac{\mathbb{P}\left(A_j \cap B\right)}{\mathbb{P}(B)} = \frac{\mathbb{P}\left(B|A_j\right)\mathbb{P}\left(A_j\right)}{\mathbb{P}\left(B \cap \left(\cup_1^n A_i\right)\right)} = \frac{\mathbb{P}\left(B|A_j\right)\mathbb{P}\left(A_j\right)}{\sum_{i=1}^n \mathbb{P}\left(B|A_i\right)\mathbb{P}\left(A_i\right)}$$

Problem 6

Let B denote the event that the result shows different numbers.

1. If A denotes the event that at least one die shows a 6, then, $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|}{|B|} = \frac{10}{36-6} = \frac{1}{3}$
2. If A denotes the event that the sum of the numbers is 8 then $A \cap B = \{(3,5),(5,3),(2,6),(6,2)\} \rightarrow$ $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|}{|B|} = \frac{4}{30} = \frac{2}{15}$

Problem 7

Let E denote the event 'she chooses the right answer', and D denote 'she knows the right answer'

$$\begin{aligned} P(D|E) =& \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} \text{ (by Bayes' Formula)} \\ =& \frac{p}{p + \frac{1}{m}(1-p)} = \frac{p}{p(1-\frac{1}{m}) + \frac{1}{m}} \rightarrow 1 \text{ as m} \rightarrow \infty \end{aligned}$$

Does this result make sense intuitively?

Problem 8

1. $\Pi_{i=1}^n (1 - p_i)$
2. $1 - \Pi_{i=1}^n (1 - p_i)$
3. $\sum_{i=1}^n p_i \Pi_{k \neq i} \Big(1 - p_k\Big)$
4.
5.
6. $1 - \Pi_{i=1}^n p_i$