SOLUTIONS OR HINTS

HW3 Problem 4. $\Omega = \{\text{subsets of size } 3 \text{ of } \{1,2,\dots,7\}\}$, $|\Omega| = \binom{7}{3}$

$X : \Omega \to \{3,4,5,6,7\}$ $\quad X(abc) = \max\{a,b,c\}$ $\quad X(236) = 6.$

$\mathbb{P}(X=3) = \mathbb{P}(123) = \frac{1}{|\Omega|}$

$\mathbb{P}(X=4) = \mathbb{P}(\{124, 134, 234\}) = \frac{3}{|\Omega|} = \frac{\binom{3}{2}}{|\Omega|}$

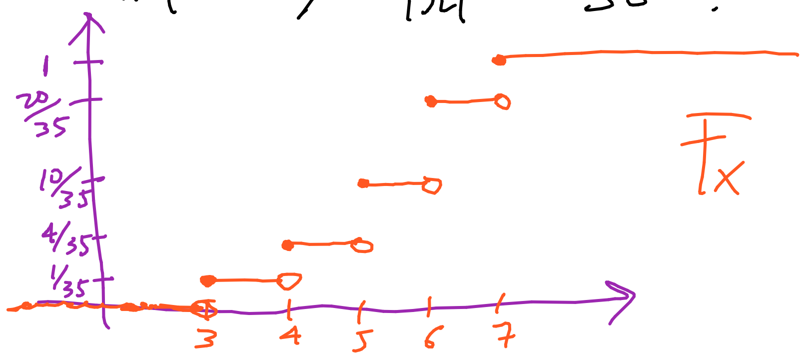
$\mathbb{P}(X=5) = \mathbb{P}(\{125, 135, 145, 235, 245, 345\}) = \frac{\binom{4}{2}}{|\Omega|} = \frac{6}{|\Omega|}$

$\mathbb{P}(X=6) = \frac{\binom{5}{2}}{|\Omega|} = \frac{10}{35}$ $\qquad \mathbb{P}(X=7) = \frac{\binom{6}{2}}{|\Omega|} = \frac{15}{35}.$

$F(3) = \frac{1}{35}$, $\quad F(4) = \frac{4}{35}$

$F(5) = \frac{10}{35}$, $\quad F(6) = \frac{20}{35}$

$F(7) = 1.$



HW4 Problem 7 $\Omega = \{w_1 w_2 w_3 : w_i \in \{H,T\}\}$

Let $X = \#$ of heads and $Y = \#$ of tails. We want $\mathbb{E}(X-Y)$.

Note that $X, Y \in \{0,1,2,3\}$, but $X - Y \in \{-3,-1,1,3\}$ why?

$\mathbb{P}(X-Y=3) = \mathbb{P}(X=3, Y=0) = \mathbb{P}(HHH) = p^3$

$\mathbb{P}(X-Y=1) = \mathbb{P}(X=2, Y=1) = \mathbb{P}(\{HHT, HTH, THH\}) = 3p^2(1-p)$

$\mathbb{P}(X-Y=-1) = \mathbb{P}(X=1, Y=2) = \mathbb{P}(\{HTT, THT, TTH\}) = 3p(1-p)^2$

$\mathbb{P}(X-Y=-3) = \mathbb{P}(X=0, Y=3) = \mathbb{P}(TTT) = (1-p)^3$, Thus

$$\mathbb{E}(X-Y) = 3 \times p^3 + 1 \times 3p^2(1-p) + (-1) \times 3p(1-p)^2 + (-3) \times (1-p)^3$$

$$= 3p^2(p + (1-p)) - 3(1-p)^2(p + (1-p))$$

$$= 3(p^2 - (1-p)^2) \times 1 \qquad = 3(2p-1)$$

Is there any faster way to compute $\mathbb{E}(X-Y)$?

Problem 6: Let D_1, D_2 be results of the dice, so $X = D_1 D_2$

Note that $X \in \{1,2,3,4,5,6,8,9,10,12,15,16,18,20,24,25,30,36\}$

- $\mathbb{P}(X=1) = \mathbb{P}(D_1=1, D_2=1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
- $\mathbb{P}(X=2) = \mathbb{P}(D_1=1, D_2=2) + \mathbb{P}(D_1=2, D_2=1) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}$
- $\mathbb{P}(X=3) = \mathbb{P}((1,3)) + \mathbb{P}((3,1)) = 1/18$
- $\mathbb{P}(X=4) = \mathbb{P}(\{(1,4),(4,1),(2,2)\}) = 3/36 = \frac{1}{12}$
- $\mathbb{P}(X=5) = 1/18$
- $\mathbb{P}(X=6) = \mathbb{P}(\{(1,6),(2,3),(3,2),(6,1)\}) = 4/36 = 1/9$

$\vdots$

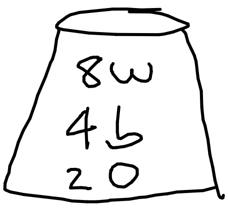
- $\mathbb{P}(X=12) = \mathbb{P}(\{(2,6),(3,4),(4,3),(6,2)\}) = 1/9$

$\vdots$

Finish it!

Problem 5: $\Omega = \{\text{pairs of balls from 14 balls}\}$, $|\Omega| = \binom{14}{2} = 91$

Let X be the earnings, so $X \in \{-2,-1,0,1,2,4\}$



$X(ww) = -2$ $\quad X(oo) = 0$ $\quad X(wo) = -1$

$X(wb) = 1$ $\quad X(bb) = 4$ $\quad X(bo) = 2$

we want $\mathbb{E}(X) = -2f(-2) - 1f(-1) + 0f(0) + 1f(1) + 2f(2) + 4f(4)$

- $f(-2) = \mathbb{P}(X=-2) = \binom{8}{2}/|\Omega| = 28/91$
- $f(-1) = \mathbb{P}(X=-1) = 8 \times 2/|\Omega| = 16/91$
- $f(1) = \mathbb{P}(X=1) = 8 \times 4/|\Omega| = 32/91$ Analogously

$f(2) = 8/91$, $f(4) = 6/91$, $f(0) = 1/91$, thus

$\mathbb{E}X = 0$.

Problem 8: 1st Solution: Let X = 1st time a black ball is selected

so $X \sim Geo(p)$, with $p = \frac{M}{M+N}$ why?

thus, $\mathbb{P}(X=i) = p(1-p)^{i-1}$ for $i = 1,2,3,\dots$ and we want:

$$\mathbb{P}(X \geq k) = \sum_{i=k}^{\infty} \mathbb{P}(X=i) = p \sum_{i=k}^{\infty} (1-p)^{i-1} = p \sum_{i=0}^{\infty} (1-p)^{i+k-1}$$

$$= p(1-p)^{k-1} \sum_{i=0}^{\infty} (1-p)^i = (1-p)^{k-1} = \left(\frac{N}{M+N}\right)^{k-1}.$$

2nd Solution: If we declare that getting a black ball is success and getting a white one is failure, then the prob. that at least $\underline{\underline{k}}$ trials are necessary to obtain a success is equal to the prob. that the first $\underline{\underline{k-1}}$ trials are all failures. So $\mathbb{P}(X \geq k) = \underbrace{\mathbb{P}(fail) \times \cdots \times \mathbb{P}(fail)}_{k-1 \text{ times}} = (1-p)^{k-1}$.

Section 3.2. Problem 2(a)

X, Y independent, both with p.m.f. $f(x) = 2^{-x}$, $x = 1,2,\dots$

We are interested in $F_Z(x)$, where $Z = \min\{X,Y\} = \begin{cases} X & \text{if } X \leq Y \\ Y & \text{if } X > Y \end{cases}$

the key observation is $Z > x \iff (X > x \text{ and } Y > x)$, so

$$\mathbb{P}(Z \leq x) = 1 - \mathbb{P}(Z > x) = 1 - \mathbb{P}(\{X > x\} \cap \{Y > x\})$$

$$= 1 - \mathbb{P}(X > x)\mathbb{P}(Y > x) \quad \text{by independence}$$

$$= 1 - 2^{-x} \cdot 2^{-x} = 1 - 4^{-x} \quad \text{for } x = 1,2,\dots$$

Since $\mathbb{P}(Y > x) = \mathbb{P}(X > x) = \sum_{i=x+1}^{\infty} \mathbb{P}(X=i)$

$$= \sum_{i=x+1}^{\infty} 2^{-i} = 2^{-x}.$$