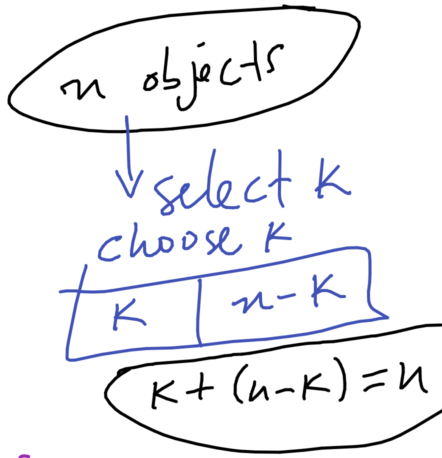
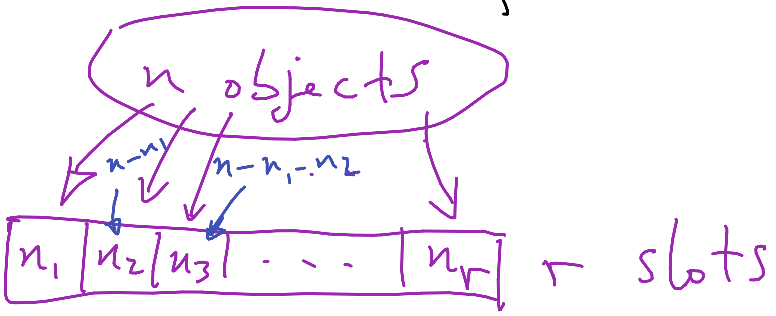
LECTURE # 4 (04/05/2021)

Recall that $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$



In general the multinomial coefficient



$$n_1 + n_2 + \dots + n_r = n$$

The number of ways to divide a set of n elements into r subsets of $n_1, n_2, \dots, n_r$ elements, where $n_1 + \dots + n_r = n$ is

$$\binom{n}{n_1, \dots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\dots-n_{r-1}}{n_r}$$

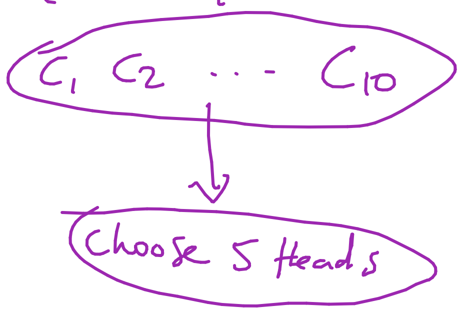
$$= \frac{n!}{n_1!\, n_2! \cdots n_r!} \quad \text{(by simplifying)}$$

Ex 2.10: A fair coin is tossed 10 times.

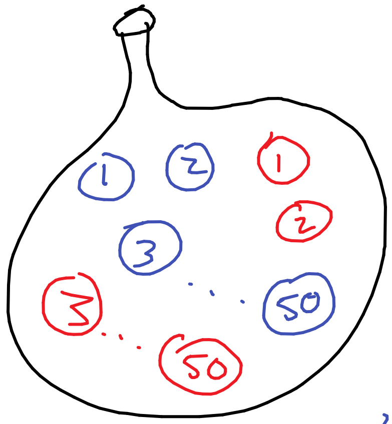
$$\mathbb{P}(\underbrace{\text{exactly 5 Heads}}) = \frac{|E|}{|S|} = \frac{\binom{10}{5}}{2^{10}} \approx 0.2461$$

$S = \{(c_1, c_2, \dots, c_{10}) : c_i \in \{H, T\}\ \forall i\} \Rightarrow |S| = 2^{10}$

$|E| = \#\text{ good outcomes} = \binom{10}{5}$



Ex 2.11: We have a bag that contains 100 balls, 50 red, 50 blue. Select 5 balls at random.



$\mathbb{P}(3 \text{ blue and } 2 \text{ red}) = \dfrac{\binom{50}{3}\binom{50}{2}}{\binom{100}{5}}$

Assume that we select all 5 balls at the same time $\Rightarrow$ $S = \{\text{subsets of size } 5 \text{ among } 100 \text{ balls}\}$

so $|S| = \binom{100}{5}$

$\#\text{ways to choose 3 blue} = \binom{50}{3}$

$\#\text{ways to choose 2 red} = \binom{50}{2}$

Note that we could choose

$S = \{(c_1, c_2, c_3, c_4, c_5) : c_1 \in \{100 \text{ balls}\},$
$c_2 \in \{100 \text{ balls}\} - \{c_1\}, \quad c_3 \in \{100 \text{ balls}\} - \{c_1, c_2\}$
$c_4 \quad \ldots \quad c_5 \ldots \}$

$|S| = 100 \times 99 \times 98 \times 97 \times 96$

TASK: $|E| = ?$

Remark: Sometimes, there is no unique choice of the set S.

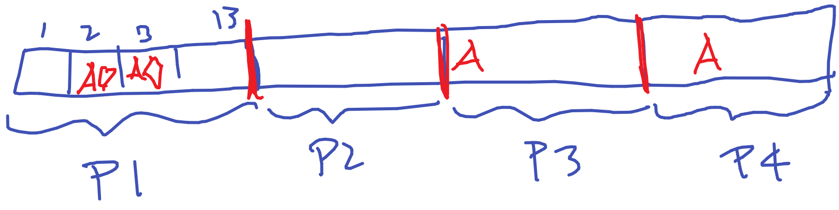
Ex 2.12: Shuffle a standard deck of 52 cards and deal 13 cards to each of the 4 players.

$\mathbb{P}(\text{each player gets an Ace})$?

Work on Solution #1: $S = \{\text{permutations of } 52 \text{ cards}\}$

$|S| = 52!$ and $\text{prob} = \dfrac{13^4 \cdot 4! \cdot 48!}{52!}$

Solution #2:



let the 1st 13 cards go to the 1st player, the 2nd 13 cards go to the 2nd player, etc.

Consider

$S = \{$positions of the 4 Aces among the 52 slots for the shuffled cards of the deck$\}$

$$|S| = \binom{52}{4}$$

$\#$ good outcomes $= 13 \times 13 \times 13 \times 13 = 13^4$

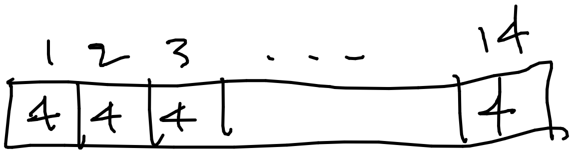
1st Ace 2nd Ace

$$\text{prob} = \frac{13^4}{\binom{52}{4}}.$$

observe $= \dfrac{13^4 \cdot 4! \cdot 48!}{52!}$

Ex 2-14:

We have 14 rooms and 4 colors, W, B, G, Y. Each room is painted at random with one of the four colors.

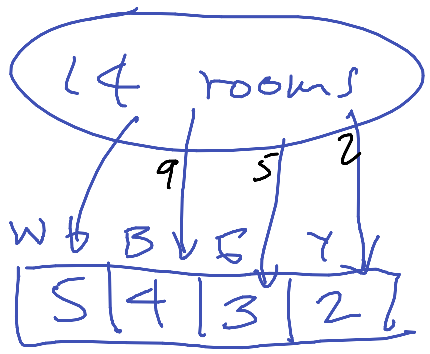


$$\mathbb{P}(5W, 4B, 3G, 2Y) = ?$$

$S = \{$paintings of 14 rooms with 4 colors$\}$

$$|S| = 4 \times 4 \times \cdots \times 4 = 4^{14}.$$

$$\text{prob} = \frac{\binom{14}{5,4,3,2}}{|S|} = \frac{\frac{14!}{5!\,4!\,3!\,2!}}{4^{14}}$$



$5 + 4 + 3 + 2 = 14$

Observation: $|E| = \binom{14}{5} \times \binom{9}{4} \times \binom{5}{3} \times \binom{2}{2}$

$$= \frac{14!}{5! \cdot 4! \cdot 3! \cdot 2!}$$

Task: Work on Example 2.15 and Problems on page 9.

§3 Axioms of Probability

Recall that we are studying probability spaces, i.e. triples $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\text{outcomes}\}$ (it could be infinite), we call Ω the sample space.

$\mathcal{F} = \{\text{events}\} \subseteq \mathcal{P}(\Omega)$ we need $\mathcal{F}$ to have certain "nice properties", or σ-algebra properties:

(P1) $\phi \in \mathcal{F}$.

(P2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (complement of event A)

(P3) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.