Math 21B-0U1, Fall 2020
October 23, 2020

TAKEHOME MIDTERM 1
Due: Today 10/23 at 18:10

NAME(use CAPITAL letters, *first name first*):______________________________

NAME(sign):________________________

ID#:__________________

HONOR STATEMENT: By signing this paper, I hereby declare that I solved this exam by my own, without any external collaboration (like friends, internet solutions, etc). If needed, I am allowed to use our lecture notes only. I understand that the main purpose of this exam is to show how much I have learned in this course, holding myself to a high standard of academic integrity, and that suspected misconduct on this exam will be reported to the Office of Student Support and Judicial Affairs and, if established, will result in disciplinary sanctions up through Dismissal from the University and a grade penalty up to a grade of 'F' for the course.

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided, or answer it in a separate paper and sign that paper; it is optional to print this exam. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. If you are using any of the problems in the textbook, then you have to solve it.

To deliver: Submit your solutions on Canvas/Gradescope, as you did with the HW1. Sign and submit the honor statement (write it down by hand, if needed).

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. $\langle 25pts \rangle$ Consider the functions

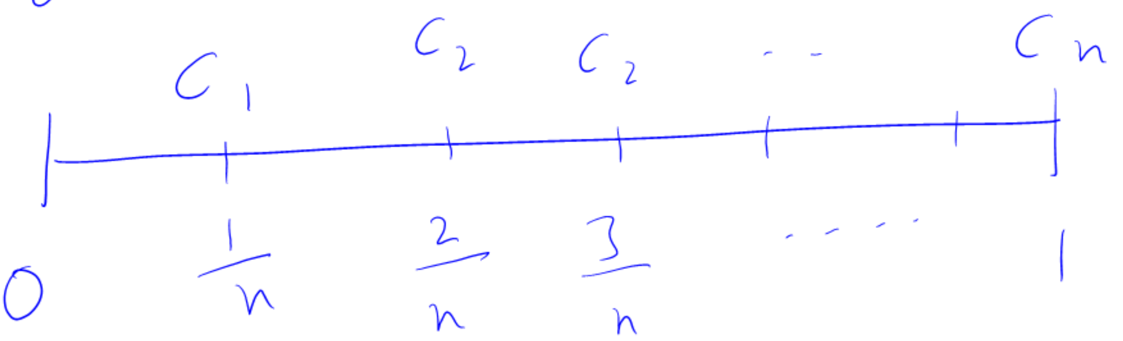
$$f(x) = e^x \qquad \text{and} \qquad g(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Use Riemann sums (and *not* the FTC) to calculate either $\int_0^1 f(x)\,dx$ or $\int_0^1 g(x)\,dx$, but NOT both.

Hint: If you consider the function f, use the formula $\sum_{k=1}^{n} a^k = \frac{a^{n+1} - a}{a-1}$, for $a \neq 1$.

(ii) $\int_0^1 g(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} g(c_i)\,\Delta x_i$

Choose the following partition & c_i's



i.e. $\Delta x_i = \frac{1}{n}$ for every n

Plugging in these values and simplifying, we get

Comment: g is a good polynomial approximation of f around 0, so $\int_0^1 f(x)\,dx \approx \int_0^1 g(x)\,dx$.

$$\int_0^1 g(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left(1 + \frac{k}{n} + \frac{1}{2}\left(\frac{k^2}{n^2}\right) + \frac{1}{6}\left(\frac{k^3}{n^3}\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} 1 + \frac{1}{n^2}\sum_{k=1}^{\infty} k + \frac{1}{2n^3}\sum_{k=1}^{n} k^2 + \frac{1}{6n^4}\sum_{k=1}^{n} n^3\right)$$

$$= \lim_{n\to\infty} \frac{n}{n} + \frac{1}{n^2}\frac{n(n+1)}{2} + \frac{1}{2n^3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{6n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$$

$$= \frac{24+12+4+1}{24} = \frac{41}{24}$$

(ii) $f(x) = e^x$

using the same partition and c_i as g we get

$$\int_0^1 e^x\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{(k/n)}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left(e^{\frac{1}{n}}\right)^k$$

$$= \lim_{n\to\infty} \frac{1}{n} \frac{e^{\frac{1}{n}(n+1)} - e^{\frac{1}{n}}}{e^{\frac{1}{n}} - 1}$$

$$= \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n\left(e^{\frac{1}{n}}-1\right)}\left(e^{\frac{n}{n}} - 1\right)$$

$$= (e-1)\lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n\left(e^{\frac{1}{n}}-1\right)}$$

$$\lim_{n\to\infty} \frac{1}{n\left(e^{\frac{1}{n}}-1\right)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{e^{\frac{1}{n}}-1}$$

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$$= \lim_{n\to\infty} \frac{-\cancel{\frac{1}{n^2}}}{e^{\frac{1}{n}}\left(-\cancel{\frac{1}{n^2}}\right)} = \frac{1}{1} = 1$$

$$\therefore \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n\left(e^{\frac{1}{n}}-1\right)} = \frac{e^0}{1} = 1$$

$$\therefore \int_0^1 e^x \, dx = e - 1$$

2. Determine whether the following statements are True or False. Justify your answers.

   (a) The following function is positive for all values of x:

$$f(x) = \frac{d^2}{dx^2}\int_0^x \left(\int_1^{\sin t} \sqrt{1+y^4}\,dy\right) dt$$

$$f(x) = \frac{d}{dx}\frac{d}{dx}\int_0^x \int_1^{\sin t} \sqrt{1+y^4}\,dy\,dt$$

$$= \int_0^{\sin x} \sqrt{1+y^4}\,dy \quad \text{(FTC)}$$

$$= \sqrt{1+\sin^4 x}\,\frac{d}{dx}\sin x \quad \text{(LEIBNIZ)}$$

$$= \cos x\sqrt{1+\sin^4 x}$$

$\cos x < 0$ for $\pi < x < \frac{3\pi}{2}$

$\therefore$ FALSE

   (b) $\langle 13pts\rangle$

$$\int_0^1 e^x \cos x\,dx \le e - 1.$$

$$\cos x \le 1 \quad \forall x$$

$$\Rightarrow e^x \cos x \le e^x \quad \forall x$$

$$\Rightarrow \int_0^1 e^x \cos x\,dx \le \int_0^1 e^x\,dx$$

$$= e^x\Big|_0^1 = e - 1$$

$\therefore$ TRUE

3. Compute the following integrals.

(a) $\langle 12pts \rangle \int \frac{u\,du}{\sqrt{1-u^4}}$

let $y = u^2$ then $dy = 2u\,du$ $\Rightarrow \frac{dy}{2} = u\,du$

$$\therefore \int \frac{u\,du}{\sqrt{1-u^4}} = \frac{1}{2}\int \frac{dy}{\sqrt{1-y^2}}$$

$$= \frac{1}{2}\sin^{-1}(y) + C = \frac{1}{2}\sin^{-1}(u^2) + C$$

(b) $\langle 13pts \rangle$

$$\int_0^4 \left|\sqrt{x} - 1\right| dx$$

$\sqrt{x} - 1 \geq 0 \Rightarrow \sqrt{x} \geq 1 \Rightarrow x \geq 1$

$\sqrt{x} - 1 < 0 \Rightarrow \sqrt{x} < 1 \Rightarrow x < 1$

$$\therefore \int_0^4 \left|\sqrt{x} - 1\right| dx = \int_0^1 \left|\sqrt{x} - 1\right| dx + \int_1^4 \left|\sqrt{x} - 1\right| dx$$

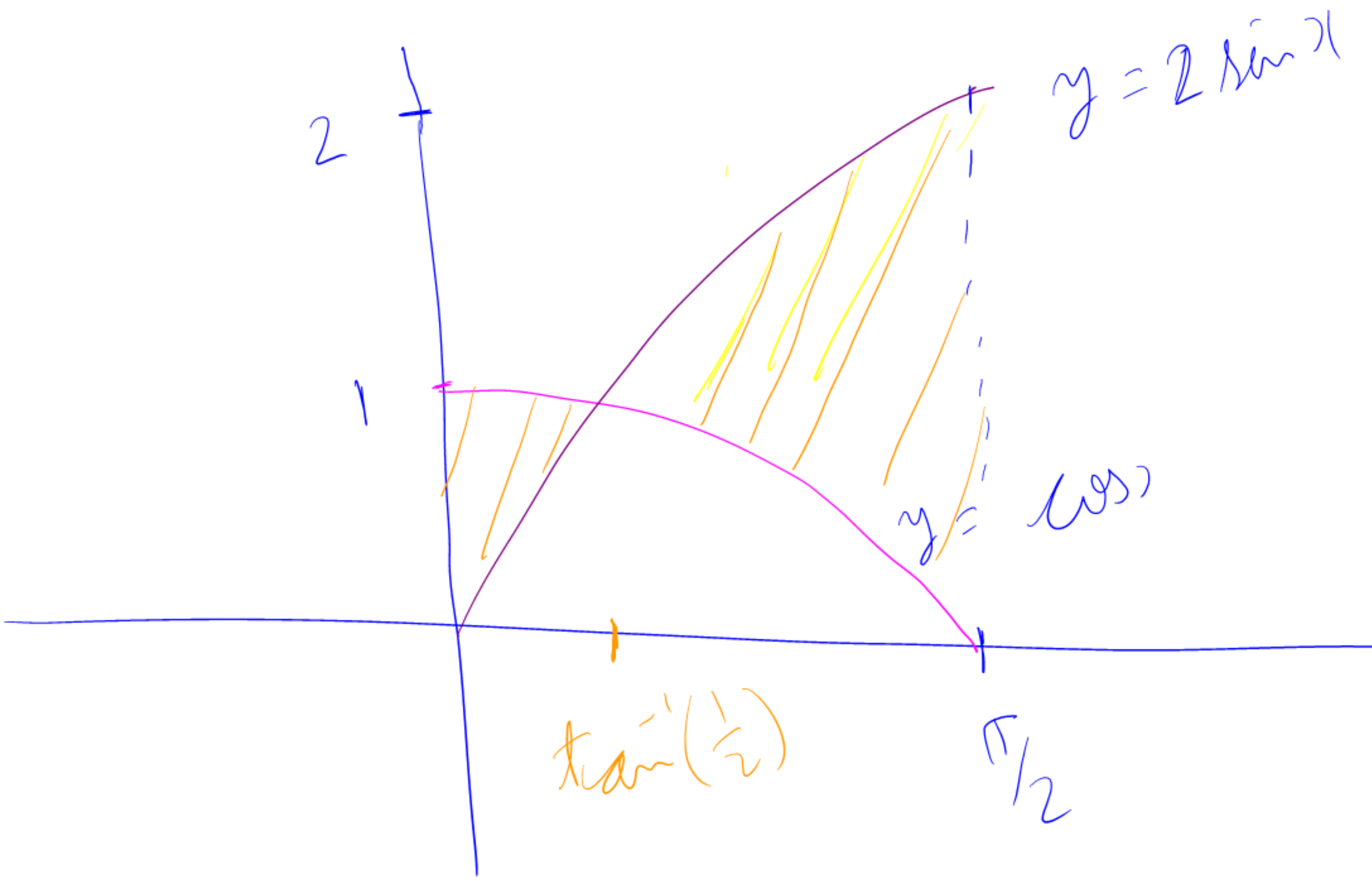
$$= \int_0^1 1 - \sqrt{x}\,dx + \int_1^4 \sqrt{x} - 1\,dx$$

$$= \left(x - \frac{2}{3}x^{3/2}\right)\Big|_0^1 + \left(\frac{2}{3}x^{3/2} - x\right)\Big|_1^4$$

$$= \left(1 - \frac{2}{3}\right) - 0 + \left(\frac{2}{3}(8) - 4\right) - \left(\frac{2}{3} - 1\right)$$

$$= -2 + \cancel{6}^{\,2}\left(\frac{2}{\cancel{3}}\right) = 2$$

4(a) PICTURE FOR 4(a)



4. (a) $\langle 13pts \rangle$ Find the area of the region bounded by the curves $y = 2\sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.

(b) $\langle 12pts \rangle$ Find the values of the constant c such that the area of the region bounded by the parabolas $x = y^2 - c^2$ and $x = c^2 - y^2$ is 576.