Math 21B, Fall 2020
December 9, 2020

TAKEHOME FINAL (sample)
Duration: 2 hours

NAME(use CAPITAL letters, *first name first*):__________________________________

NAME(sign):____________________________

ID#:____________________

HONOR STATEMENT: By signing this paper, I hereby declare that I solved this exam by my own, without any external collaboration (like friends, internet solutions, etc). If needed, I am allowed to use our lecture notes only. I understand that the main purpose of this exam is to show how much I have learned in this course, holding myself to a high standard of academic integrity, and that suspected misconduct on this exam will be reported to the Office of Student Support and Judicial Affairs and, if established, will result in disciplinary sanctions up through Dismissal from the University and a grade penalty up to a grade of 'F' for the course.

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided, or answer it in a separate paper and sign that paper; it is optional to print this exam. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. If you are using any of the problems in the textbook, then you have to solve it.

To deliver: Submit your solutions on Canvas/Gradescope, as you did with the MT1. Sign and submit the honor statement (write it down by hand, if needed).

Make sure that you have a total of 7 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	200 pts

x	$e^{-x^2/2}$
0	1
1/4	$e^{-1/32}$
1/2	$e^{-1/8}$
3/4	$e^{-9/32}$
1	$e^{-1/2}$

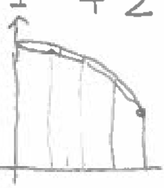
1. Throughout this problem, consider the integral $\int_0^1 e^{-x^2/2}\,dx$.

(a) Partition the interval $[0, 1]$ into 4 subintervals of equal length, and let the evaluation points c_k be the *left* endpoints. Write down (but do not evaluate) the resulting approximating (Riemann) sum to the integral.

$$\frac{1}{4}\left[1 + e^{-1/32} + e^{-1/8} + e^{-9/32}\right]$$

(b) Use the trapezoidal rule with 4 subintervals to write down an approximating expression for the above integral. (Do not evaluate this expression.)

$$\frac{1}{8}\left(1 + 2e^{-1/32} + 2e^{-1/8} + 2e^{-9/32} + e^{-1/2}\right)$$



(c) For each approximation (in (a) and (b)), determine whether it overestimates or underestimates the integral.

$f(x) = e^{-x^2/2}$

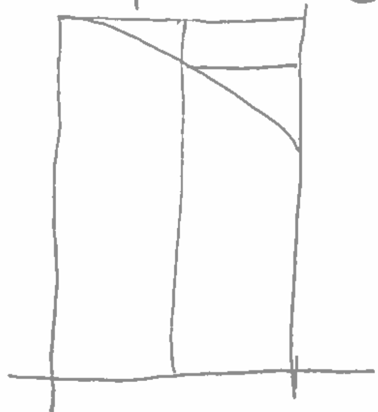
$f'(x) = -xe^{-x^2/2} \leq 0$ f is decreasing, so the approx. in (a) overestimates

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = (x^2-1)e^{-x^2/2} \leq 0$ for x in $[0,1]$

f is concave down, so the approx. in (b) underestimates.

(d) Assume that you increase the number of subintervals in (a) to 8, but keep everything else the same. Is the approximating sum with 8 intervals larger or smaller than the one in (a) with 4 intervals?

Every rectangle in (a) is replaced by two rectangles: the sum with 8 intervals is smaller.



2

2. Compute the following indefinite integrals.

(a) $\int \frac{4(x+1)}{x^2(x+2)}\,dx$

$$\frac{4(x+1)}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$* x^2,\ x=0 ;\ B=2$

$* (x+2),\ x=-2 :\ C=-1$

$* x,\ x \to \infty ;\ 0 = A + C,\ A = 1$

$$= \ln|x| - 2 \cdot \frac{1}{x} - \ln|x+2| + C$$

(b) $\int x^3 \sin(2x^2)\,dx$

$t = 2x^2 \qquad dt = 4x\,dx$

$$= \frac{1}{8}\int t \sin t\,dt$$

$u = t \quad du = dt$

$dv = \sin t\,dt \quad v = -\cos t$

$$= \frac{1}{8}\left[-t\cos t + \int \cos t\,dt\right]$$

$$= \frac{1}{8}\left[-t\cos t + \sin t\right] + C$$

$$= -\frac{x^2}{4}\cos(2x^2) + \frac{1}{8}\sin(2x^2) + C$$

3. Determine whether the two improper integrals below converge or diverge.

(a) $\int_1^{e^4} \frac{1}{x \cdot \sqrt{\ln x}}\,dx$ (Improper because $\ln x = 0$ at $x = 1$.)

$= \lim_{c \to 1+} \int_c^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx$ $t = \ln x$ $dt = \frac{1}{x}dx$

$= \lim_{c \to 1+} \int_{\ln c}^{4} \frac{1}{\sqrt{t}}\,dt$

$= \lim_{c \to 1+} 2\sqrt{t}\,\Big|_{\ln c}^{4} = \lim_{c \to 1+} \left(4 - 2\sqrt{\ln c}\right)$

$= \underline{\underline{4}}$; converges.

(b) $\int_1^{\infty} \underbrace{\sqrt{x} \cdot \frac{x^2 + 2x + 3}{4x^3 + 5x + 6}}_{f(x)}\,dx$

$g(x) = \sqrt{x} \cdot \frac{x^2}{x^3} = \frac{1}{\sqrt{x}}$

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x \cdot (x^2 + 2x + 3)}{4x^3 + 5x + 6} = \frac{1}{4}$,

so $\int_1^{\infty} f(x)\,dx$ and $\int_1^{\infty} g(x)\,dx$ conv. or div. together.

But we know that $\int_1^{\infty} g(x)\,dx$ diverges,

and so $\int_1^{\infty} f(x)\,dx$ __diverges__.

4. Determine whether the following statements are True or False and give reasons for your answers.

(a) If $a_n = \displaystyle\sum_{k=1}^{n} \frac{k}{n^2 + k^2}$, then $\lim_{n\to\infty} a_n = \ln 2$. [Hint: a_n is a Riemann sum]

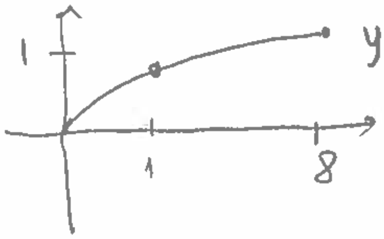
(b)

$$\int_0^1 \frac{x^2}{(5 - x^2)^{3/2}}\, dx \leq \frac{1}{8}.$$

(c) If a vertical rectangular plate a units long by b units wide is submerged in a fluid of weight-density w with its long edges parallel to the fluid's surface, then the force exerted by the fluid on one side of the plate is the average value of the pressure along the vertical dimension of the plate times the area of the plate.

8. Set up, but do not evaluate, the integrals for the arc length of the following curves. Also sketch roughly each of the curves (no concavity analysis necessary).

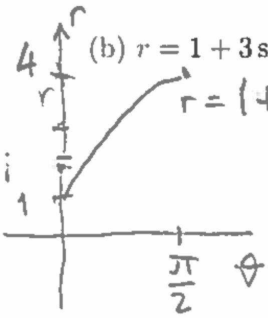
(a) $y = x^{1/3}$, $1 \le x \le 8$, in Cartesian coordinates.

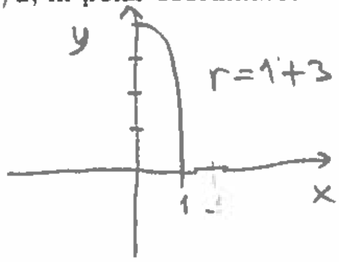


$$A.L. = \int_1^8 \sqrt{1 + \frac{1}{9} x^{-4/3}}\, dx$$

$$y' = \frac{1}{3} x^{-2/3}$$

(b) $r = 1 + 3\sin\theta$, $0 \le \theta \le \pi/2$, in polar coordinates.

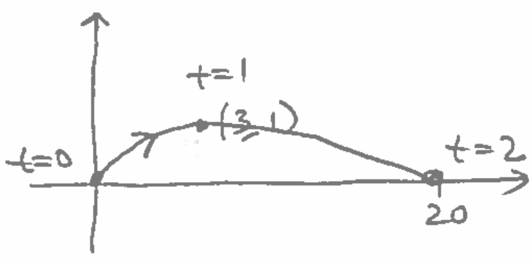




$$\frac{dr}{d\theta} = 3\cos\theta$$

$$A.L. = \int_0^{\pi/2} \sqrt{(1+3\sin\theta)^2 + 9\cos^2\theta}\, d\theta$$

(c) $x = t^2 + 2t^3$, $y = 2t - t^2$, $0 \le t \le 2$, given parametrically.



$$\frac{dx}{dt} = 2t + 6t^2 \ge 0$$

$$\frac{dy}{dt} = 2 - 2t = 2(1-t) \begin{cases} \ge 0 & \text{for } t \le 1 \\ < 0 & \text{for } t \ge 1 \end{cases}$$

$$A.L. = \int_0^2 \sqrt{(2t+6t^2)^2 + 4(1-t)^2}\, dt$$

6. Consider the curve given parametrically by $x = \cos^3 t$ and $y = \sin^3 t$, $0 \le t \le \pi/2$.
(a) Find the slope of the tangent to the curve at $t = \pi/4$.

$$\frac{dx}{dt} = -3\cos^2 t \sin t \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\tan t$$

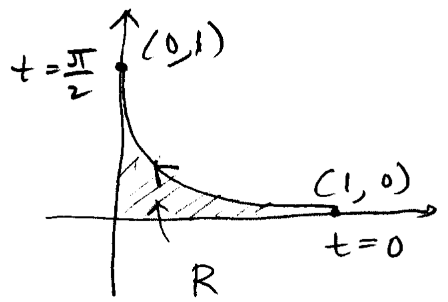
$$\frac{dy}{dt} = 3\sin^2 t \cos t$$

Slope at $t = \frac{\pi}{4}$ equals $-\tan\frac{\pi}{4} = \underline{\underline{-1}}$.

(b) Determine whether the curve is concave up or concave down. Sketch the curve.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-\tan t)}{-3\cos^2 t \sin t} = -\frac{\frac{1}{\cos^2 t}}{-3\cos^2 t \sin t} = \frac{1}{3\cos^4 t \sin t} > 0 \text{ for } t \in [0, \tfrac{\pi}{2}]$$

The curve is <u>concave up</u>.



(c) Set up, but do not evaluate, the integral for the area of the region R bounded by the curve, the x–axis and the y–axis.

$$\int_{x=0}^{x=1} y\,dx = \int_{\pi/2}^{0} \sin^3 t\,(-3\cos^2 t \sin t)\,dt$$

$$= \int_0^{\pi/2} 3\sin^4 t \cos^2 t\,dt$$

(d) Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating R around the y–axis.

$$\int_{x=0}^{x=1} 2\pi xy\,dx = 2\pi\int_0^{\pi/2} \sin^3 t \cos^3 t \cdot 3\cos^2 t \sin t\,dt$$

$$= 6\pi\int_0^{\pi/2} \sin^4 t \cos^5 t\,dt$$