Math 21B, Fall 2020
October 16, 2020

TAKEHOME MIDTERM 1 (sample)

NAME(use CAPITAL letters, *first name first*):____________________________________

NAME(sign):____________________________

ID#:____________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in either the space provided or a separate paper. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Unless directed to do so, do *not* prove any theorem or proposition seen in class, and do not evaluate complicated expressions to give the result as a fraction or a decimal number. However, if you are using any of the problems in the textbook, then you have to solve or prove it.

To deliver: Submit your solutions on Canvas/Gradescope in either pdf or jpg format. Sign and submit this first page.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

Before starting: Imagine that this is your real exam and solve it in less than 50 minutes. Problems come with solutions, so solve them on a different paper. **Avoid reading the solutions**, and compare them with yours only when you finish your simulation.

1. Suppose that f has a positive derivative for all values $x \in \mathbb{R}$ and that $f(1) = 0$. Consider

$$g(x) = \int_0^x f(t)\,dt.$$

Determine whether the following statements about the function g are True or False and give reasons for your answers.

(a) g is integrable over $[0, 1]$.

(b) g has a local maximum at $x = 1$.

(c) The graph of dg/dx crosses the x-axis at $x = 1$.

2. Throughout this problem, consider $F(x) = \int_0^x (t-1)e^{t^2}\,dt$, and restrict x to $x \geq 0$.

(a) Determine the intervals on which $y = F(x)$ is increasing and those on which it is decreasing.

$F'(x) = (x-1)e^{x^2}$

$F'(x) > 0$ when $x > 1$, so F is increasing for $x > 1$

$F'(x) < 0$ when $x < 1$, so F is decreasing for $x > 1$.

(b) Determine the intervals on which $y = F(x)$ is concave up and those on which it is concave down.

$F''(x) = e^{x^2} + (x-1)2x\,e^{x^2} = \underbrace{(2x^2-2x+1)}_{\text{always } >0}\,e^{x^2}$

always > 0, because

$D = (-2)^2 - 4\cdot 2 = -4 < 0$,

so $2x^2-2x+1$ is never 0,

and is 1 at $x=0$.

Always concave up.

(c) Determine $\lim_{x\to 0^+} \frac{F(x)+x}{x^3+5x^2}$ $= \lim_{x\to 0+} \frac{(x-1)e^{x^2}+1}{3x^2+10x} =$

$\left(\frac{0}{0}\right)$ L'Hopital $\left(\frac{0}{0}\right)$ L'Hopital

$= \lim_{x\to 0+} \frac{(2x^2-2x+1)e^{x^2}}{6x+10} = \frac{1}{10}$

3. Compute the following two antiderivatives.

(a) $\int \frac{(\ln x)^4}{3x}\,dx$ $= \frac{1}{3}\int u^4\,du = \frac{1}{15}u^5 + C$

$\ln x = u$

$= \frac{1}{15}(\ln x)^5 + C$

$\frac{1}{x}dx = du$

(b) $\int \frac{x^5}{\sqrt{x^3+4}}\,dx$ $= \int \frac{x^3 \cdot x^2}{\sqrt{x^3+4}}\,dx$

$u = x^3 + 4$ $\qquad du = 3x^2\,dx$

$x^3 = u - 4$

$= \frac{1}{3}\int \frac{(u-4)\,du}{\sqrt{u}}$

$= \frac{1}{3}\int \left(u^{1/2} - 4u^{-1/2}\right) du$

$= \frac{1}{3}\left(\frac{u^{3/2}}{3/2} - 4\,\frac{u^{1/2}}{1/2}\right) + C$

$= 2\,(x^3+4)^{3/2} - \frac{8}{3}(x^3+4)^{1/2} + C$

4. Compute the following two areas.

(a) Area under the graph of $y = \dfrac{\sin x \cdot (1+\cos x)}{(3+\cos x)^2}$ on the interval $[0, \pi/2]$. Explain first why the function is never negative on this interval.

$\sin x \geq 0$ on $[0, \pi/2]$;

$1+\cos x \geq 0$ everywhere;

the denominator is a square.

$$\int_0^{\pi/2} \frac{\sin x\,(1+\cos x)}{(3+\cos x)^2}\,dx = -\int_4^3 \frac{u-2}{u^2}\,du$$

$u = 3+\cos x$

$du = -\sin x\,dx$

$1+\cos x = u-2$

x	u
0	4
$\pi/2$	3

$$= \int_3^4 \left(\frac{1}{u} - 2u^{-2}\right) du$$

$$= \ln u + 2u^{-1} \Big|_3^4$$

$$= \ln 4 - \ln 3 + \frac{2}{4} - \frac{2}{3}$$

$$= \ln\frac{4}{3} - \frac{1}{6}$$

(b) Area of the bounded region enclosed by the graphs of $y = x^3 + x^2$ and of $y = x^3 - x^2 + 4x$. Give the result as a simple fraction.

Intersections: $\cancel{x^3} + x^2 = \cancel{x^3} - x^2 + 4x$

$2x^2 = 4x \qquad x^2 - 2x = 0$

$x = 0, 2$

At $x = 1$: $x^3 + x^2 = 2$

$x^3 - x^2 + 4x = 4$

$y = x^3 - x^2 + 4x$ is the upper funct.

$$\int_0^2 (-x^2 + 4x - x^2)\,dx = \int_0^2 (4x - 2x^2)\,dx$$

$$= 2x^2 - \frac{2x^3}{3}\Big|_0^2 = 8 - \frac{16}{3} = \frac{8}{3}.$$