Math 21B, Fall 2020
November 13, 2020

TAKEHOME MIDTERM 2 (sample)

NAME(use CAPITAL letters, *first name first*):____

NAME(sign):________________________________

ID#:_______________________

HONOR STATEMENT: By signing this paper, I hereby declare that I solved this exam by my own, without any external collaboration (like friends, internet solutions, etc). If needed, I am allowed to use our lecture notes only. I understand that the main purpose of this exam is to show how much I have learned in this course, holding myself to a high standard of academic integrity, and that suspected misconduct on this exam will be reported to the Office of Student Support and Judicial Affairs and, if established, will result in disciplinary sanctions up through Dismissal from the University and a grade penalty up to a grade of 'F' for the course.

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided, or answer it in a separate paper and sign that paper; it is optional to print this exam. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. If you are using any of the problems in the textbook, then you have to solve it.

To deliver: Submit your solutions on Canvas/Gradescope, as you did with the MT1. Sign and submit the honor statement (write it down by hand, if needed).

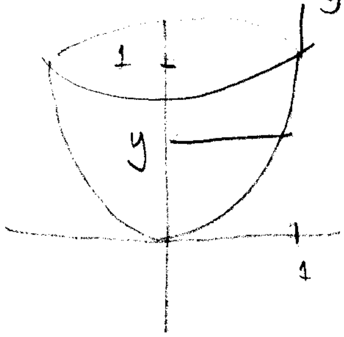
Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

Before starting: Imagine that this is your real exam and solve it in less than 50 minutes. Problems come with solutions, so solve them on a different paper. **Avoid reading the solutions**, and compare them with yours only when you finish your simulation.

1. A reservoir is obtained by rotating the curve $y = x^3$, $0 \le x \le 1$, around the y axis. (The length units are meters.) It is completely filled by liquid of density ρ. Leave ρ and g arbitrary in your answers.

(a) Determine the work needed to pump the liquid out of the top of the reservoir.



$y = x^3$

$x = y^{1/3}$

$$\text{work} = \int_0^1 \underset{\substack{\uparrow \\ \text{distance}}}{(1-y)} \cdot \underbrace{g \cdot \rho \cdot \overbrace{\pi \left(y^{1/3}\right)^2 dy}^{\text{Vol.}}}_{\text{force}}$$

$$= \pi \rho g \int_0^1 (1-y) \cdot y^{2/3}\, dy$$

$$= \pi \rho g \int_0^1 \left(y^{2/3} - y^{5/3}\right) dy$$

$$= \pi \rho g \left(\frac{y^{5/3}}{5/3} - \frac{y^{8/3}}{8/3} \right) \Big|_0^1$$

$$= \pi \rho g \left(\frac{3}{5} - \frac{3}{8} \right) = 3\pi \rho g \frac{3}{40}$$

$$= \underline{\underline{\frac{9}{40} \pi \rho g}}$$

(b) Determine the work needed to lift the full reservoir by 1 meter.

This equals mass $\cdot\, g$ = $\rho g \cdot$ volume

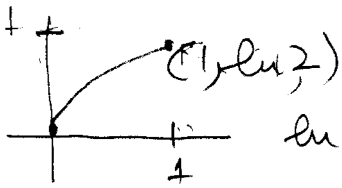
$$= \rho g \int_0^1 \pi \left(y^{1/3}\right)^2 dy = \pi \rho g \int_0^1 y^{2/3} dy$$

$$= \pi \rho g \cdot \frac{3}{5} = \underline{\underline{\frac{3}{5} \pi \rho g}}$$

2

2. Consider the curve given as the graph of the function $y = \ln(x+1)$, for $0 \le x \le 1$. Write down, but *do not compute* the integrals for quantities specified below.
(a) The arc length of this curve.

$$\frac{dy}{dx} = \frac{1}{x+1}$$



$\ln 2 < 1$

$$A.L. = \int_0^1 \sqrt{1 + \left(\frac{1}{x+1}\right)^2}\, dx$$

(b) The surface area of the surface obtained by revolution of this curve around the x axis.

$$S.A. = 2\pi \int_0^1 \ln(x+1) \sqrt{1 + \left(\frac{1}{1+x}\right)^2}\, dx$$

$$\left(= \int_0^1 2\pi y\, ds \right)$$

(c) The surface area of the surface obtained by revolution of this curve around the line $y = 1$. (Explain why this line does not intersect the curve!)

$$\left(= \int_0^1 2\pi (1-y)\, ds \right)$$

$$= 2\pi \int_0^1 (1 - \ln(x+1)) \sqrt{1 + \left(\frac{1}{x+1}\right)^2}\, dx$$

3

2. Compute the following two definite integrals.

(a) $\int_0^1 \ln(x^2+1)\,dx$

$u = \ln(x^2+1)$ $\quad du = \dfrac{2x}{x^2+1}\,dx$

$dv = dx$ $\quad v = x$

$$= x\ln(x^2+1)\Big|_0^1 - 2\int_0^1 \frac{x^2}{x^2+1}\,dx$$

$$= \ln 2 - 2\int_0^1 \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \ln 2 - 2\left[x - \arctan x\right]_0^1 = \ln 2 - 2 + \frac{\pi}{4}$$

(b) $\int_{-1}^{5} \frac{1}{x^2+2x+10}\,dx$ $\quad = \int_{-1}^{5} \dfrac{1}{(x+1)^2+9}\,dx$

$x+1 = 3u$

$dx = 3\,du$

x	u
-1	0
5	2

$$= \int_0^2 \frac{3\,du}{9u^2+9} = \frac{1}{3}\int_0^2 \frac{1}{u^2+1}\,du$$

$$= \frac{1}{3}\arctan u\Big|_0^2 = \frac{1}{3}\arctan 2$$

4. Determine whether the following statements are True or False and give reasons for your answers.

(a) There are at least two different curves $y = y(x)$ whose arc length for $1 \leq x \leq 9$ is

$$L = \int_1^9 \sqrt{1 + \frac{1}{4x}}\, dx$$

(b)

$$\int_{-\pi}^{\pi} \sin^{2020}(x)\, dx = -\frac{2\pi}{2021}.$$

(c) Every integer $n \geq 2$ satisfies the inequality $\ln n < \sum_{k=1}^{n-1} \frac{1}{k}$.