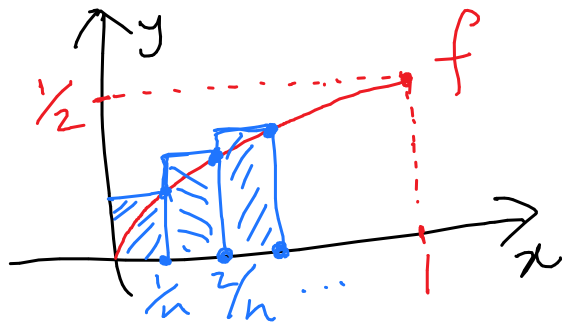
TAKEHOME FINAL (sample)

Solutions to Problem 4.

ⓐ $a_n = \sum_{k=1}^{n} \frac{k}{n^2+k^2} = \sum_{k=1}^{n} \frac{k/n}{1+(k/n)^2} \cdot \frac{1}{n}$

$= \sum_{k=1}^{n} f(k/n) \cdot \frac{1}{n}$,

where $f(x) = \frac{x}{1+x^2}$,



so a_n is a Riemann sum for $\int_0^1 f(x)\,dx$, thus

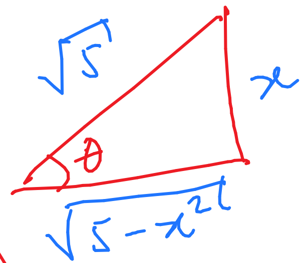
$$\lim_{n\to\infty} a_n = \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2|\Big|_0^1 = \frac{1}{2}\ln 2$$

$\neq \ln 2$, so the statement is FALSE.

ⓑ TRUE: We can use trigonometric substitution:

$x = \sqrt{5}\sin\theta$ for $\theta \in [-\frac{\pi}{2}, \pi/2]$,

$dx = \sqrt{5}\cos\theta\, d\theta$, $5 - x^2 = 5\cos^2\theta$.



$\cos\theta \geq 0$

$$\int \frac{x^2\,dx}{(5-x^2)^{3/2}} = \int \frac{(\sqrt{5}\sin\theta)^2 \sqrt{5}\cos\theta\, d\theta}{(\sqrt{5\cos^2\theta})^3}$$

$$= \int \frac{\sqrt{5}^3 \sin^2\theta\cos\theta}{\sqrt{5}^3 \cos^3\theta}\,d\theta = \int \frac{(1-\cos^2\theta)\cos\theta}{\cos^3\theta}\,d\theta$$

$$= \int \frac{d\theta}{\cos^2\theta} - \int d\theta = \tan\theta - \theta + C$$

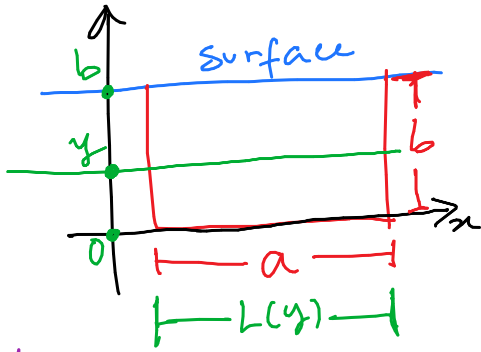
$$= \frac{x}{\sqrt{5-x^2}} - \arcsin\left(\frac{x}{\sqrt{5}}\right) + C,$$ so

$$\int_0^1 \frac{x^2\,dx}{(5-x^2)^{3/2}} = \left[\frac{x}{\sqrt{5-x^2}} - \arcsin\left(\frac{x}{\sqrt{5}}\right)\right]\Big|_0^1$$

$$= \left[\frac{1}{2} - \arcsin\left(\frac{1}{\sqrt{5}}\right)\right] - 0 \simeq 0.037 \le \frac{1}{8}.$$

Remark: There is an easier way to justify that (b) is true. Can you figure it out?

(c) TRUE: • The Area of the plate is ab.



• The pressure at level y is $P(y) = wy$, so the average of the pressure is

$$av(P) = \frac{1}{b-0}\int_0^b P(y)\,dy = \frac{1}{b}\int_0^b wy\,dy$$

$$= \frac{w}{b}\,\frac{y^2}{2}\Big|_0^b = \frac{w}{b}\cdot\frac{b^2}{2} = \frac{wb}{2}.$$

• Finally, the force exerced by the fluid is

$$F = \int_0^b w\cdot\left(\begin{matrix}\text{strip}\\ \text{depth}\end{matrix}\right)\cdot L(y)\,dy = \int_0^a w\cdot y\cdot a\,dy$$

$$= wa\,\frac{y^2}{2}\Big|_0^b = wa\,\frac{b^2}{2} = \left(\frac{wb}{2}\right)\cdot(ab)$$

$$= av(P)\times(\text{Area of plate}),$$

as claimed. #